MAE Preliminary Examination

Mathematics Section

Monday, November 16, 2026, 1:00pm-3:30pm

Your Name

THREE OF THE FOUR PROBLEMS WILL BE GRADED **Select the 3 problems you have worked on, to be graded:**	**points**
Problem:	/10
Problem:	/10
Problem:	/10
Total	/30

Please give your answers/work in the space provided

Explain your work/steps clearly

Linear Algebra: Problem 1 [10 points]:

Let $A \in \mathbb{R}^{4\times 4}$ have eigenvalues $\{1, 2, 2, 2\}$.

(i) What is the characteristic polynomial of A?

(ii) Assume that there are three linearly independent eigenvectors associated with the eigenvalue 2. What is the minimal polynomial of A? Explain why.

(iii) Under the assumption in part (ii), express $A^{1/2}$ as a polynomial in A; that is, find coefficients a, b such that

$$A^{1/2} = aA + bI.$$

(iv) Show that the eigenvalues of

$$U = (AA^\top)^{-1/2}A$$

have modulus one.

Hint: First reason that U is invertible, and then note that U is orthogonal (verify).

Workspace for Problem 1: Explain your reasoning/work here.

Solution

(i) The characteristic polynomial is

$$p_A(\lambda) = (\lambda - 1)(\lambda - 2)^3.$$

(ii) The eigenvalue 2 has algebraic multiplicity 3 and, by assumption, geometric multiplicity 3. Hence all Jordan blocks associated with 2 have size one. The eigenvalue 1 is simple, so its Jordan block also has size one. Therefore A is diagonalizable and its minimal polynomial is

$$m_A(\lambda) = (\lambda - 1)(\lambda - 2).$$

(iii) Since A is diagonalizable and its only distinct eigenvalues are 1 and 2, it suffices to find a linear polynomial $q(\lambda) = a\lambda + b$ satisfying

$$q(1) = 1, \qquad q(2) = \sqrt{2}.$$

Thus

$$a + b = 1, \qquad 2a + b = \sqrt{2},$$

so

$$a = \sqrt{2} - 1, \qquad b = 2 - \sqrt{2}.$$

Hence

$$\boxed{A^{1/2} = (\sqrt{2} - 1)A + (2 - \sqrt{2})I.}$$

(iv) Since A is invertible, $AA^\top$ is positive definite and $(AA^\top)^{-1/2}$ is well defined. Moreover,

$$UU^\top = (AA^\top)^{-1/2}AA^\top(AA^\top)^{-1/2} = I.$$

Thus U is orthogonal. If $Uv = \lambda v$ for a nonzero (possibly complex) vector v, then

$$\|v\| = \|Uv\| = \|\lambda v\| = |\lambda|\,\|v\|,$$

and therefore

$$\boxed{|\lambda| = 1.}$$

Workspace for Problem 1: Explain your reasoning/work here.

Differential Equations: Problem 2 [10 points]:

Consider the coupled differential equations

$$\dot{u} = v^2 - u, \qquad \dot{v} = \tfrac{1}{2}\, v\,(3 - 2u - v^2).$$

Do the following:

1. Write the system in state-space form $\dot{x} = f(x)$, with $x = (x_1, x_2)^T$.
2. Find all equilibrium states, $x_{\mathrm{eq},i}$.
3. Determine the linearized dynamics about each equilibrium.
4. Determine the eigenvalues of each linearized system, and classify the equilibria. For any foci, determine the sense of rotation (clockwise/counterclockwise).
5. Sketch the phase portrait.

Workspace for Problem 2: Explain your reasoning/work here.

Solution:

1. With $x_1 = u$, $x_2 = v$,

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = f(x) = \begin{pmatrix} x_2^2 - x_1 \\ \tfrac{1}{2}\, x_2\,(3 - 2x_1 - x_2^2) \end{pmatrix}.$$

2. Equilibria satisfy $x_1 = x_2^2$ and $x_2(3 - 2x_1 - x_2^2) = 0$. Substituting the first into the second,

$$x_2\,(3 - 3x_2^2) = 0 \quad \Longrightarrow \quad x_2 \in \{0, \pm 1\}, \qquad x_1 = x_2^2 \in \{0, 1\}.$$

Hence the equilibrium points are

$$(0,0), \qquad (1,1), \qquad (1,-1).$$

3. The Jacobian is

$$J(x_1, x_2) = \begin{pmatrix} -1 & 2x_2 \\ -x_2 & \tfrac{1}{2}\,(3 - 2x_1 - 3x_2^2) \end{pmatrix},$$

and the linearized dynamics about $x_{\mathrm{eq},i}$ are $\dot{\xi} = J(x_{\mathrm{eq},i})\,\xi$, with $\xi = x - x_{\mathrm{eq},i}$:

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}, \qquad J(1,1) = \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix}, \qquad J(1,-1) = \begin{pmatrix} -1 & -2 \\ 1 & -1 \end{pmatrix}.$$

4. *At* $(0,0)$*:* the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = \frac{3}{2}$, of opposite sign. Hence $(0,0)$ is a saddle point. The stable eigendirection is the x_1-axis and the unstable one is the x_2-axis.

At $(1, \pm 1)$*:* in both cases

$$\det(\lambda I - J) = (\lambda + 1)^2 + 2 = \lambda^2 + 2\lambda + 3 = 0 \quad \Longrightarrow \quad \lambda_{1,2} = -1 \pm i\sqrt{2}.$$

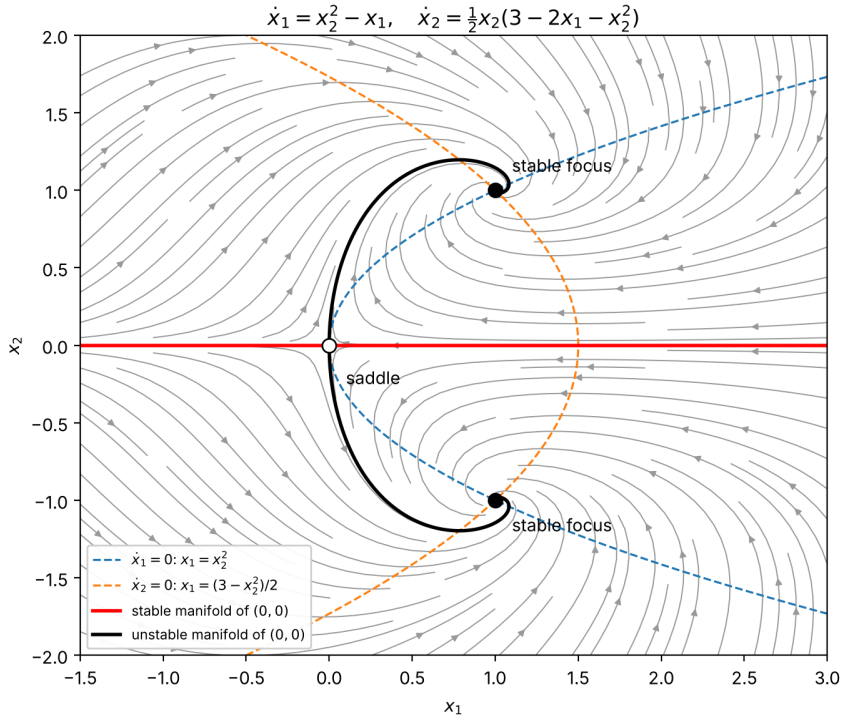
Complex eigenvalues with negative real part, so $(1,1)$ and $(1,-1)$ are both stable foci (stable spirals).

Sense of rotation: at $(1,1)$, the displacement $\xi = (0,1)^T$ gives $\dot{\xi} = J\xi = (2,-1)^T$: a point directly above the equilibrium moves to the right, so trajectories rotate *clockwise*. At $(1,-1)$, $\xi = (0,1)^T$ gives $\dot{\xi} = (-2,-1)^T$, so trajectories rotate *counterclockwise*. The two foci rotate in opposite directions. This is consistent with the symmetry of the vector field: under $(x_1, x_2) \mapsto (x_1, -x_2)$, f_1 is even in x_2 and f_2 is odd. The system is therefore invariant under reflection about the x_1-axis, and reflection reverses orientation.

5. The nullclines are

$$\dot{x}_1 = 0 \iff x_1 = x_2^2, \qquad \dot{x}_2 = 0 \iff x_2 = 0 \text{ or } x_1 = \tfrac{1}{2}(3 - x_2^2).$$

The x_1-axis is invariant, and on it $\dot{x}_1 = -x_1$, so it is the stable manifold of the saddle. The two branches of the unstable manifold leave the origin vertically, one into each half-plane, and spiral into $(1,1)$ and $(1,-1)$, respectively. (Optional: with the Dulac function $B = 1/x_2$, $\nabla \cdot (Bf) = -1/x_2 - x_2 < 0$ for $x_2 > 0$, so there are no periodic orbits in the upper half-plane. Every trajectory with $x_2(0) > 0$ converges to $(1,1)$, and by symmetry every trajectory with $x_2(0) < 0$ converges to $(1,-1)$.)


$\dot{x}_1 = x_2^2 - x_1, \quad \dot{x}_2 = \frac{1}{2}x_2(3 - 2x_1 - x_2^2)$
stable focus
saddle
stable focus
$\dot{x}_1 = 0$: $x_1 = x_2^2$
$\dot{x}_2 = 0$: $x_1 = (3 - x_2^2)/2$
stable manifold of (0, 0)
unstable manifold of (0, 0)
x_1
x_2

Workspace for Problem 2: Explain your reasoning/work here.

Partial Differential Equations: Problem 3 [10 points]:

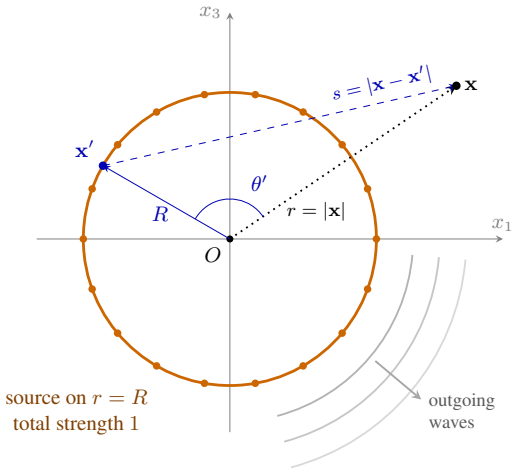
A source of unit total strength and frequency $\omega > 0$ is spread uniformly over the sphere $r = R$ in infinite three-dimensional space, where $r = |\mathbf{x}|$ (see figure). The field $\phi(\mathbf{x}, t)$ satisfies the wave equation with wave speed c,

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi - c^2\, q(\mathbf{x})\, e^{-i\omega t}, \qquad q(\mathbf{x}) = \frac{\delta(|\mathbf{x}| - R)}{4\pi R^2}, \qquad \int q\, dV = \int_0^\infty \frac{\delta(r-R)}{4\pi R^2}\, 4\pi r^2\, dr = 1,$$

and is outgoing at infinity. Here δ is the one-dimensional delta function of the radial distance r, and R is the radius of the sphere (a number, not a vector): q vanishes everywhere except on the sphere, where it is uniform.

You may use the following result: the outgoing field at $\mathbf{x}$ of a unit point source at $\mathbf{x}'$, i.e. the outgoing solution of $\nabla^2 G + k^2 G = \delta(\mathbf{x} - \mathbf{x}')$ with k as found in (i), is

$$G(\mathbf{x}, \mathbf{x}') = -\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}.$$



Source spread over the sphere $r = R$; $\mathbf{x}'$ is a point of the sphere and $\mathbf{x}$ the observer.

(i) We seek the solution that oscillates at the source frequency, $\phi = u(\mathbf{x})\, e^{-i\omega t}$. Find the equation satisfied by u, and identify k.

(ii) Using G, write $u(\mathbf{x})$ as an integral over the sphere $|\mathbf{x}'| = R$, and explain why it satisfies the equation for u and is outgoing. (*Hint:* the source is distributed on the surface of the sphere only, so the volume integral reduces to a surface integral.)

(iii) Evaluate the integral for $r > R$ and show that $u = F\, u_{\text{free}}$, where $u_{\text{free}} = -e^{ikr}/(4\pi r)$ is the field of a unit point source at the origin and

$$F = \frac{\sin kR}{kR}.$$

(*Hint:* use spherical coordinates for $\mathbf{x}'$ with the polar axis along $\mathbf{x}$, so that the area element of the sphere is $dS' = R^2 \sin\theta'\, d\theta'\, d\psi'$, and change the integration variable from θ' to $s = |\mathbf{x} - \mathbf{x}'|$, where $s^2 = r^2 + R^2 - 2rR\cos\theta'$.)

Workspace for Problem 3: Explain your reasoning/work here.

Workspace for Problem 3: Explain your reasoning/work here.

Solution

(i) With $\phi = u\,e^{-i\omega t}$, $\phi_{tt} = -\omega^2 u\,e^{-i\omega t}$. Substituting into the wave equation and cancelling $e^{-i\omega t} \neq 0$ gives $-\omega^2 u = c^2\nabla^2 u - c^2 q(\mathbf{x})$. Dividing by c^2 and rearranging,

$$\nabla^2 u + k^2 u = q(\mathbf{x}), \qquad k = \frac{\omega}{c}$$

(the Helmholtz equation). The factor $e^{-i\omega t}$ drops out, so a solution at the source frequency exists. (Any other solution differs from it by a free wave fixed by the initial conditions, which radiates away to infinity; the time-harmonic solution is the long-time response.)

(ii) *Superposition.* Split the source into small pieces: the piece at $\mathbf{x}'$ is a point source of strength $q(\mathbf{x}')\,dV'$, whose field at $\mathbf{x}$ is $G(\mathbf{x},\mathbf{x}')\,q(\mathbf{x}')\,dV'$. The equation is linear, so the fields add:

$$u(\mathbf{x}) = \int G(\mathbf{x},\mathbf{x}')\,q(\mathbf{x}')\,dV'.$$

Check: the operator acts on $\mathbf{x}$, not on $\mathbf{x}'$, so $(\nabla^2 + k^2)u = \int \delta(\mathbf{x}-\mathbf{x}')\,q(\mathbf{x}')\,dV' = q(\mathbf{x})$; and u is outgoing because each G is.

Surface integral. With $q(\mathbf{x}') = \delta(r'-R)/(4\pi R^2)$, $r' = |\mathbf{x}'|$, and $dV' = r'^2\,dr'\,d\Omega'$,

$$u(\mathbf{x}) = \int\!\!\int_0^\infty G(\mathbf{x},\mathbf{x}')\,\frac{\delta(r'-R)}{4\pi R^2}\,r'^2\,dr'\,d\Omega'.$$

The delta function sets $r' = R$ ($\int_0^\infty f(r')\,\delta(r'-R)\,dr' = f(R)$): $r'^2 \to R^2$ and $\mathbf{x}'$ becomes a point of the sphere. Only the angles remain, and $R^2 d\Omega' = dS'$ is the area element of the sphere. With $s = |\mathbf{x}-\mathbf{x}'|$,

$$u(\mathbf{x}) = \frac{1}{4\pi R^2}\oint_{|\mathbf{x}'|=R} G\,dS' = -\frac{1}{4\pi R^2}\oint_{|\mathbf{x}'|=R}\frac{e^{iks}}{4\pi s}\,dS',$$

i.e. $u = \frac{1}{4\pi}\int G\,d\Omega'$ is the average of the point-source field over the sphere.

(iii) With the polar axis along $\mathbf{x}$ (allowed, since the source is spherically symmetric), the angle between $\mathbf{x}$ and $\mathbf{x}'$ is θ' and $dS' = R^2\sin\theta'\,d\theta'\,d\psi'$; the integrand does not depend on ψ', which gives a factor 2π. From $s^2 = r^2 + R^2 - 2rR\cos\theta'$, $s\,ds = rR\sin\theta'\,d\theta'$; for $r > R$, s runs from $r-R$ ($\theta' = 0$) to $r+R$ ($\theta' = \pi$). Then

$$\oint \frac{e^{iks}}{s}\,dS' = 2\pi R^2\int_{r-R}^{r+R}\frac{e^{iks}}{s}\,\frac{s\,ds}{rR} = \frac{2\pi R}{r}\,\frac{e^{ik(r+R)} - e^{ik(r-R)}}{ik} = \frac{4\pi R}{r}\,\frac{\sin kR}{k}\,e^{ikr},$$

using $e^{ik(r+R)} - e^{ik(r-R)} = e^{ikr}\left(e^{ikR} - e^{-ikR}\right) = 2i\sin(kR)\,e^{ikr}$. Hence

$$u = -\frac{1}{4\pi R^2}\,\frac{1}{4\pi}\,\frac{4\pi R}{r}\,\frac{\sin kR}{k}\,e^{ikr} = \frac{\sin kR}{kR}\left(-\frac{e^{ikr}}{4\pi r}\right), \qquad \boxed{u = F\,u_{\text{free}}, \quad F = \frac{\sin kR}{kR} \quad (r > R).}$$

Workspace for Problem 3: Explain your reasoning/work here.

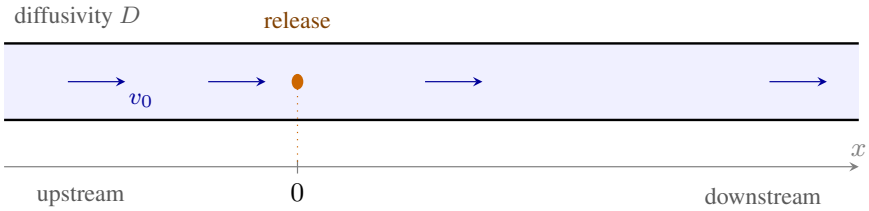
Partial Differential Equations: Problem 4 [10 points]:

Heat (or a dye) is carried along a very long pipe by a uniform flow of speed $v_0 \geq 0$ while it diffuses with diffusivity $D > 0$ (see figure). Its temperature (or concentration) $T(x,t)$ on $-\infty < x < \infty$ satisfies, for $t > 0$,

$$\frac{\partial T}{\partial t} + v_0 \frac{\partial T}{\partial x} = D \frac{\partial^2 T}{\partial x^2}.$$

An amount Q is released at $x = 0$ at $t = 0$: $T(x,0) = Q\,\delta(x)$, with $T \to 0$ as $|x| \to \infty$. Use the Fourier transform $\hat{T}(k,t) = \int_{-\infty}^{\infty} T(x,t)\, e^{-ikx}\, dx$, with inverse $T(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{T}(k,t)\, e^{ikx}\, dk$. You may use, for $\gamma > 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma k^2}\, e^{ikx}\, dk = \frac{e^{-x^2/(4\gamma)}}{\sqrt{4\pi\gamma}}.$$



A pipe with uniform flow v_0; material is released at $x = 0$.

(i) Fourier-transform the PDE and show that

$$\frac{d\hat{T}}{dt} = -\left(Dk^2 + ikv_0\right)\hat{T}, \qquad \hat{T}(k,0) = Q.$$

Solve for $\hat{T}(k,t)$.

(ii) For $v_0 = 0$, invert the transform using the given integral to find $T(x,t)$.

(iii) For $v_0 > 0$, show that

$$T(x,t) = \frac{Q}{\sqrt{4\pi D t}}\, \exp\left[-\frac{(x - v_0 t)^2}{4Dt}\right].$$

(*Hint:* combine $e^{-ikv_0 t}$ with e^{ikx}.) What does the convection term do to the cloud?

Workspace for Problem 4: Explain your reasoning/work here.

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Fourier Transform Reference Tables

$$F(k) = \hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx, \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk.$$

This is the convention used in Problem 4: the factor $1/(2\pi)$ occurs in the inverse transform. Below, $G = \hat{g}$, $a, \alpha, \beta > 0$, $b, k_0 \in \mathbb{R}$, and $i^2 = -1$.

Standard transform pairs

$f(x)$	$F(k)$	Description
$e^{-\alpha x^2}$	$\sqrt{\frac{\pi}{\alpha}} e^{-k^2/(4\alpha)}$	Gaussian
$\frac{e^{-x^2/(4\beta)}}{\sqrt{4\pi\beta}}$	$e^{-\beta k^2}$	Heat kernel
$\delta(x)$	1	Dirac delta
$\delta(x-b)$	e^{-ikb}	Shifted delta
1	$2\pi\delta(k)$	Constant
$e^{ik_0 x}$	$2\pi\delta(k-k_0)$	Complex exponential
$\cos(k_0 x)$	$\pi[\delta(k-k_0)+\delta(k+k_0)]$	Cosine
$\sin(k_0 x)$	$\frac{\pi}{i}[\delta(k-k_0)-\delta(k+k_0)]$	Sine
$\frac{2\alpha}{x^2+\alpha^2}$	$2\pi e^{-\alpha\lvert k\rvert}$	Lorentzian
$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+k^2}$	Two-sided exponential
$e^{-ax}H(x)$	$\frac{1}{a+ik}$	One-sided exponential

Fourier Transform Properties

Function	Fourier transform	Property
$Af(x) + Bg(x)$	$AF(k) + BG(k)$	Linearity
$\dfrac{\partial f(x,t)}{\partial t}$	$\dfrac{\partial F(k,t)}{\partial t}$	Time derivative
$\dfrac{\partial f}{\partial x}$	$ikF(k)$	First spatial derivative
$\dfrac{\partial^2 f}{\partial x^2}$	$-k^2F(k)$	Second spatial derivative
$\dfrac{\partial^n f}{\partial x^n}$	$(ik)^n F(k)$	nth spatial derivative
$xf(x)$	$i\dfrac{dF}{dk}$	Multiplication by x
$x^n f(x)$	$i^n \dfrac{d^n F}{dk^n}$	Multiplication by x^n
$f(x-b)$	$e^{-ikb}F(k)$	Spatial shift
$e^{ik_0 x}f(x)$	$F(k-k_0)$	Frequency shift
$f(cx), \quad c \in \mathbb{R} \setminus \{0\}$	$\dfrac{1}{\lvert c\rvert}F\left(\dfrac{k}{c}\right)$	Scaling
$f(-x)$	$F(-k)$	Reflection
$\overline{f(x)}$	$\overline{F(-k)}$	Complex conjugation
$(f * g)(x) = \displaystyle\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi$	$F(k)G(k)$	Convolution
$f(x)g(x)$	$\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(q)G(k-q)\,dq$	Product

Here A, B are constants and n is a nonnegative integer. The transform is taken in x, with t held fixed. Assume sufficient integrability, smoothness, and decay for each identity; spatial integration by parts requires the boundary terms to vanish. For real-valued f, $F(-k) = \overline{F(k)}$.

Integral identities

$$F(0) = \int_{-\infty}^{\infty} f(x)\,dx, \qquad f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\,dk$$

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\overline{G(k)}\,dk.$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(k)|^2\,dk.$$

The point-evaluation formulas require the relevant integrals to exist and inversion to hold at the point in question (in particular, continuity at $x = 0$ for the displayed formula for $f(0)$). The inner-product and energy identities hold for square-integrable functions.

Solution

(i) With $\widehat{T_x} = ik\hat{T}$ and $\widehat{T_{xx}} = -k^2\hat{T}$, the PDE becomes, for each k, an ODE in t:

$$\frac{d\hat{T}}{dt} + ikv_0\hat{T} = -Dk^2\hat{T} \quad \Longrightarrow \quad \frac{d\hat{T}}{dt} = -\left(Dk^2 + ikv_0\right)\hat{T}.$$

The initial condition transforms to $\hat{T}(k,0) = \int Q\,\delta(x)\,e^{-ikx}\,dx = Q$. Hence

$$\hat{T}(k,t) = Q\,e^{-Dk^2t}\,e^{-ikv_0t}.$$

(ii) For $v_0 = 0$, with $\gamma = Dt$ in the given integral,

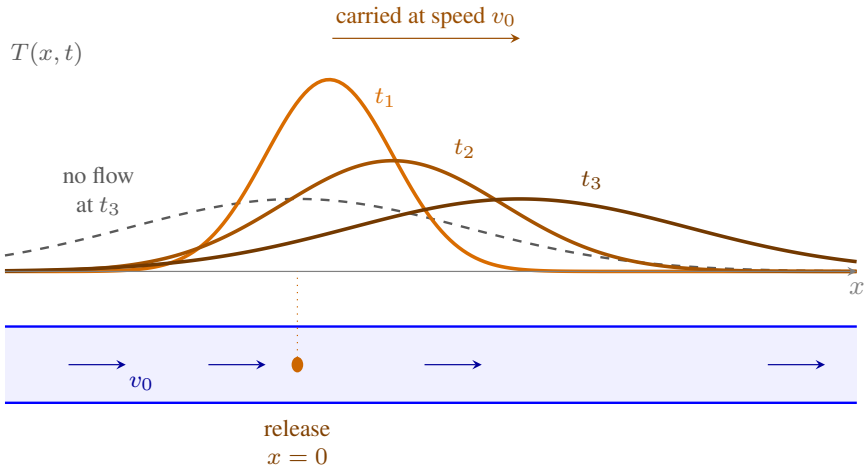
$$T(x,t) = \frac{Q}{2\pi}\int e^{-Dk^2t}e^{ikx}\,dk = \frac{Q}{\sqrt{4\pi Dt}}\,e^{-x^2/(4Dt)}\,:$$

a Gaussian centered at the release point, spreading with standard deviation $\sqrt{2Dt}$; the total amount $\int T\,dx = Q$ is conserved.

(iii) For $v_0 > 0$, $e^{-ikv_0t}e^{ikx} = e^{ik(x-v_0t)}$, so the inverse transform is the integral of (ii) evaluated at $x - v_0t$ (shift theorem):

$$\boxed{T(x,t) = \frac{Q}{\sqrt{4\pi Dt}}\,\exp\left[-\frac{(x-v_0t)^2}{4Dt}\right].}$$

Effect of convection. The cloud is carried downstream at speed v_0 while it spreads exactly as without flow: the factor e^{-ikv_0t} is a pure phase, so it translates the solution without changing its shape. Equivalently, in a frame moving with the flow, $\xi = x - v_0t$, the equation becomes the plain diffusion equation.



$T(x,t)$ at three times $t_1 < t_2 < t_3$. With flow (solid) the cloud is carried downstream at speed v_0 while it spreads exactly as without flow (dashed, shown at t_3), which stays centered on the release point.