MAE Preliminary Examination

Mathematics Section

Tentative: Monday, April 14, 1:00pm-3:30pm

Your Name

THREE PROBLEMS WILL BE GRADED **Select the 3 problems you've worked, to be graded:**	**points**
Problem:	$/10$
Problem:	$/10$
Problem:	$/10$
Total	$/30$

Please give your answers/work in the space provided

Explain your work/steps clearly

Calculators are allowed but not computers

Linear Algebra: Problem 1 [10 points]:

Let:
D be a $n \times n$ diagonal matrix with all entries on the diagonal being positive,
M be an $n \times n$ symmetric and positive definite matrix, and
N be an $n \times n$ (not necessarily symmetric nor positive definite) matrix, that is invertible for simplicity.

With D, M, N as above, and U any $n \times n$ orthogonal matrix, show the following:

1. $\mathrm{trace}(DU) \leq \mathrm{trace}(D)$. [3 points]
2. $\mathrm{trace}(MU) \leq \mathrm{trace}(M)$. [4 points]
3. $\mathrm{trace}(NU) \leq \mathrm{trace}((NN^T)^{1/2})$. [3 points]

Note: you can use knowledge of the validity of 1. to prove 2. and of 2. to prove 3.

Hints:
– Recall that the trace, $\mathrm{trace}(A)$, is the sum of all diagonal elements of a square matrix A.
– Recall that an $n \times n$ orthogonal matrix U is a matrix such that $UU^T = U^TU = I$, with I the identity matrix.
– Finally, recall that any matrix A has a factorization $A = R\Theta$ where $R = (AA^T)^{1/2}$ and Θ is orthogonal.

Workspace for Problem 1: Explain your reasoning/work here.

Solution:

1. Note that since $UU^T = I$, for every row i, $\sum_j |U_{ij}|^2 = 1$. Hence, all elements in U have modulus < 1. Therefore, $\mathrm{trace}(DU) = \sum_i d_i U_{ii} \leq \sum_i d_i |U_{ii}| \leq \sum_i d_i = \mathrm{trace}(D)$.
2. Any symmetric matrix can be diagonalized by an orthogonal transformation, namely, M can be written as

$$M = VDV^T$$

for some V that is an orthogonal matrix. Moreover, D is diagonal, and since $M > 0$, all entries of D on the diagonal are positive, and these are the eigenvalues of M. In fact, $\mathrm{trace}(M) = \mathrm{trace}(DV^TV) = \mathrm{trace}(D)$. Then,

$$\begin{aligned}
\mathrm{trace}(MU) &\leq \mathrm{trace}(VDV^TU) \\
&= \mathrm{trace}(DV^TUV), \text{ since } \mathrm{trace}(AB) = \mathrm{trace}(BA), \\
&\leq \mathrm{trace}(D), \text{ since } V^TUV \text{ is orthogonal and invoking statement 1.,} \\
&= \mathrm{trace}(M).
\end{aligned}$$

3. Simply write $N = (NN^T)^{1/2}\Theta$ and observe that

$$\begin{aligned}
\mathrm{trace}(NU) &\leq \mathrm{trace}((NN^T)^{1/2}\Theta U) \\
&= \mathrm{trace}((NN^T)^{1/2}), \text{ since } \Theta U \text{ is orthogonal.}
\end{aligned}$$

Workspace for Problem 1: Explain your reasoning/work here.

Ordinary Differential Equations: Problem 2 [10 points]:

Determine the phase portrait of the 2nd order differential equation $\ddot{x} = 4x - 3\dot{x} - x^3$. To do this:
i) Express the dynamics as a first-order differential equation with states

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

ii) Find all points of equilibrium.
iii) Linearize the dynamics near each point of equilibrium and ascertain the nature of the particular equilibrium (i.e., saddle point, focus, etc.).
iv) Sketch the phase portrait.

Solution:
The first-order differential equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 4x_1 - 3x_2 - x_1^3 \end{bmatrix}$$

The point of equilibrium $e_1 = (2, 0)$, $e_2 = (-2, 0)$ and $e_3 = (0, 0)$

Linearization:

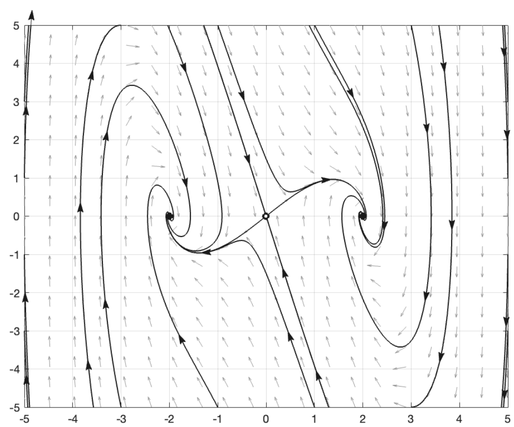
$$A = \begin{bmatrix} 0 & 1 \\ 4 - 3x_1^2 & -3 \end{bmatrix}$$

and then compute the eigenvalues and the eigenvector of the linearized matrices

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -8 & -3 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}.$$

The eigenvalues of $A_{1,2}$ (negative real part) and the eigenvalues of A_3 are $\lambda_{1,2} = 1, -4$.

The phase portrait is shown as follows



Workspace for Problem 2: Explain your reasoning/work here.

Partial Differential Equations: Problem 3 [10 points]:

Solve the following problem for $u(x,t)$ using the Green's function method.

$$u_t = ku_{xx} \text{ for } 0 \le x < \infty \ , \ t \ge 0$$

subject to the initial and boundary conditions:

$$\begin{aligned} u(x,0) &= f(x) \\ u_x(0,t) &= g(t) \end{aligned}$$

(a) Find the Green's function $G(x,\xi,t,\tau)$ that satisfies appropriate initial and boundary conditions for the above problem.
(b) Find the solution $u(x,t)$ as an integral of the Green's function and the initial and boundary conditions.
(c) Find the solution $u(0,t)$ as a function of x for $f(x) = 0$ and $g(t) = -1$.

Hint: the one-dimensional infinite domain $-\infty < x < \infty$ Green's function for the heat equation is

$$G^{\infty}(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi k(t-\tau)}} e^{-\frac{(x-\xi)^2}{4k(t-\tau)}}$$

Solution:
(a) The Green's function for the corresponding homogeneous boundary condition $G_x(0,t;\xi,\tau) = 0$ can be found by placing an image of equal strength at $(-\xi,t)$, yielding

$$G(x,t;\xi,\tau) = G^{\infty}(x,t;\xi,\tau) + G^{\infty}(x,t;-\xi,\tau) = \frac{1}{\sqrt{4\pi k(t-\tau)}} e^{-\frac{(x-\xi)^2}{4k(t-\tau)}} + \frac{1}{\sqrt{4\pi k(t-\tau)}} e^{-\frac{(x+\xi)^2}{4k(t-\tau)}}$$

You can easily check that it does satisfy the boundary condition $G_x(0,t;\xi,\tau) = 0$ and decays to zero value at infinite distance.

(b) According to the Green's formula, the solution of the problem is then

$$u(x,t) = -k\int_0^t g(\tau)G(x,t;0,\tau)d\tau + \int_0^{\infty} f(\xi)G(x,t;\xi,0)d\xi$$

$$u(x,t) = -k\int_0^t g(\tau)\frac{e^{-\frac{x^2}{k(t-\tau)}}}{\sqrt{\pi k(t-\tau)}}d\tau + \int_0^{\infty} f(\xi)\frac{e^{-\frac{(x-\xi)^2}{4kt}} + e^{-\frac{(x+\xi)^2}{4kt}}}{\sqrt{4\pi kt}}$$

(c) Evaluating the above solution at $x = 0$ with $f(x) = 0$ and $g(t) = -1$ and defining $p = t - \tau$ gives

$$u(0,t) = k\int_0^t \frac{1}{\sqrt{\pi k(t-\tau)}}d\tau = -k\int_t^0 \frac{1}{\sqrt{\pi kp}}dp = k\int_0^t \frac{1}{\sqrt{\pi kp}}dp = \sqrt{\frac{4kt}{\pi}}$$

Workspace for Problem 3: Explain your reasoning/work here.

Partial Differential Equations: Problem 4 [10 points]:

Solve the wave equation

$$u_{tt} - a^2 u_{xx} \;=\; 0 \;\text{ for } \; -\infty < x < \infty \;\;,\; t \geq 0$$

subject to the initial conditions:

$$u(x,0) \;=\; 0$$
$$u_t(x,0) \;=\; e^{ikx}$$

using:

(a) d'Alembert's formula;

(b) Fourier transform, showing all the steps of your derivation and proving that you match the solution in (a).

Workspace for Problem 4: Explain your reasoning/work here.

Solution:

(a) We solve the Cauchy problem using d'Alembert's formula

$$u(x,t) = f(x-at) + g(x+at)$$

The solution with $u(x,0) = U(x)$ and $u_t(x,0) = V(x)$ is

$$u(x,t) = \frac{1}{2}\Big[U(x-at) + U(x+at)\Big] \;+\; \frac{1}{2a}\int_{x-at}^{x+at} V(s)ds$$

But here $U = 0$, so we are left with

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} V(s)ds$$

Evaluating for $V(s) = e^{iks}$,

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} e^{iks}ds \;=\; \frac{1}{2iak}\left[e^{ik(x+at)} - e^{ik(x-at)}\right]$$

which reduces to

$$\boxed{u(x,t) = \frac{e^{ikx}}{ak}\sin(akt)}$$

(b) Since x is infinite, we apply the Fourier transform in x:

$$\widehat{u}(\omega,t) \;=\; \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx$$

The FT of the governing equation is

$$\frac{d^2\widehat{u}}{dt^2} - (-a^2\omega^2)\widehat{u} \;=\; 0 \quad\rightarrow\quad \frac{d^2\widehat{u}}{dt^2} + (a\omega)^2\widehat{u} \;=\; 0$$

while the FTs of the initial conditions are:

$$\widehat{u}(\omega,0) \;=\; 0$$
$$\widehat{u}_t(\omega,0) \;=\; 2\pi\delta(\omega-k)$$

The solution to the ODE in the transformed domain is

$$\widehat{u}(\omega,t) \;=\; A(\omega)\cos(a\omega t) \;+\; B(\omega)\sin(a\omega t)$$

and its time derivative is

$$\widehat{u}_t(\omega,t) \;=\; -a\omega A(\omega)\sin(a\omega t) \;+\; a\omega B(\omega)\cos(a\omega t)$$

Application of the first initial condition requires $A(\omega) = 0$, while the second initial condition gives

$$B(\omega) \;=\; \frac{2\pi}{a\omega}\delta(\omega-k)$$

So we have

$$\widehat{u}(\omega, t) \;=\; \frac{2\pi}{a\omega}\,\delta(\omega - k)\;\sin(a\omega t)$$

Returning to the original domain,

$$u(x, t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(\omega, t)\, e^{i\omega x} d\omega$$

Inserting our expression for $\widehat{u}(\omega, t)$,

$$u(x, t) \;=\; \int_{-\infty}^{\infty} \frac{1}{a\omega}\,\delta(\omega - k)\;\sin(a\omega t) e^{i\omega x} d\omega$$

The delta function eliminates all contributions except at $\omega = k$:

$$\boxed{u(x, t) \;=\; \frac{e^{ikx}}{ak}\sin(akt)}$$

The solution is identical to that obtained using d'Alembert's formula.

Workspace for Problem 4: Explain your reasoning/work here.