MAE Preliminary Examination

Mathematics Section

Monday, April 20, 2026, 1:00pm-3:30pm

Your Name

THREE OF THE FOUR PROBLEMS WILL BE GRADED **Select the 3 problems you have worked on, to be graded:**	**points**
Problem:	$/10$
Problem:	$/10$
Problem:	$/10$
Total	$/30$

Please give your answers/work in the space provided

Explain your work/steps clearly

Linear Algebra: Problem 1 [10 points]:

Let $A, B \in \mathbb{R}^{n \times n}$ and assume that the characteristic polynomial of A is $\det(sI - A) = s^3 + s^2 + s + 1$. (Hint: one eigenvalue is -1.)

(a) Express A^{-1} as a polynomial function of A.

(b) Can A be a symmetric matrix? If not, explain why not, and if yes, produce an example.

(c) Determine the Jordan form of A?

(d) Explain why A is similar to the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}?$$

Workspace for Problem 1: Explain your reasoning/work here.

Solution

(a) By Caley-Hamilton, $A^3 + A^2 + A + I = 0$, hence $A^{-1} = -A^2 - A - I$.

(b) The roots of the characteristic polynomial of A, $\{-1, i, -i\}$, includes imaginary ones. A symmetric matrix always has real eigenvalues. Hence, A cannot be symmetric.

(c) The eigenvalues of A are as above $\{-1, \pm i\}$, hence the real Jordan form is

$$J_{A,\mathbb{R}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

and the complex, is

$$J_{A,\mathbb{C}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

Either solution is acceptable for this problem.

(d) This is always the case, since they have the same characteristic polynomial and the roots are simple. That is, they are both diagonalizable and share the same Jordan form over $\mathbb{C}$.

Workspace for Problem 1: Explain your reasoning/work here.

Differential Equations: Problem 2 [10 points]:

For the stationary nonlinear system

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2^2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

Do the following:

1. Find all equilibrium states, $x_{eq,i}$.
2. Determine the linearized dynamics.
3. Determine the eigenvalues of each linearized system, and classify the equilibria.
4. Sketch the phase portrait.

Workspace for Problem 2: Explain your reasoning/work here.

Solution:

1. Equilibria satisfy

$$x_1 - x_2^2 = 0, \qquad x_2 - x_1 x_2 = x_2 - x_2^3 = 0.$$

Thus

$$x_2 \in \{0, \pm 1\} \text{ corresponding to } x_1 = x_2^2 \in \{0, 1\}.$$

Therefore the equilibrium points are

$$(0,0), \qquad (1,1), \qquad (1,-1).$$

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2. The Jacobian matrix is

$$J(x_1, x_2) = \begin{pmatrix} \frac{\partial}{\partial x_1}(x_1 - x_2^2) & \frac{\partial}{\partial x_2}(x_1 - x_2^2) \\ \frac{\partial}{\partial x_1}(x_2 - x_1 x_2) & \frac{\partial}{\partial x_2}(x_2 - x_1 x_2) \end{pmatrix} = \begin{pmatrix} 1 & -2x_2 \\ -x_2 & 1 - x_1 \end{pmatrix}.$$

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3. Linearized dynamics & classification:

At $(0, 0)$:

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues are

$$\lambda_1 = \lambda_2 = 1.$$

Since both eigenvalues are positive, $(0,0)$ is an **unstable node** (a source).

At $(1, 1)$:

$$J(1,1) = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det \begin{pmatrix} 1-\lambda & -2 \\ -1 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2.$$

Hence

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0,$$

so the eigenvalues are

$$\lambda_1 = 2, \qquad \lambda_2 = -1.$$

Therefore $(1, 1)$ is a **saddle point**.

At $(1, -1)$:

$$J(1,-1) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2.$$

Thus the eigenvalues are again

$$\lambda_1 = 2, \qquad \lambda_2 = -1.$$

Therefore $(1, -1)$ is also a **saddle point**.

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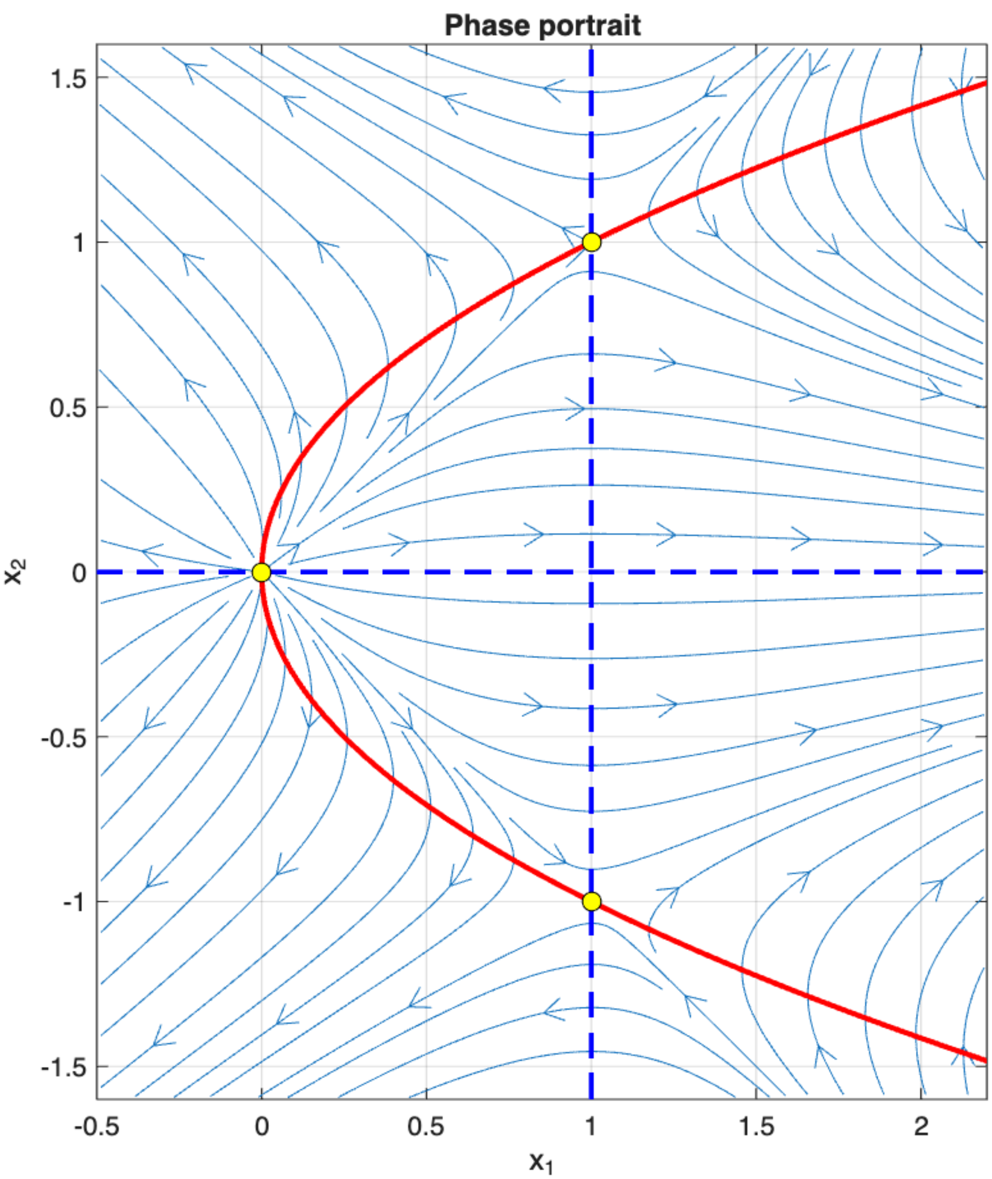
4. The phase portrait is as follows:



The curve where $\dot{x}_1 = 0$:

$$x_1 - x_2^2 = 0 \quad \iff \quad x_1 = x_2^2,$$

is a parabola drawn in red. The curves where $\dot{x}_2 = 0$:

$$x_2(1 - x_1) = 0 \quad \Longleftrightarrow \quad x_2 = 0 \quad \text{or} \quad x_1 = 1,$$

are dotted in blue.

To determine the flow direction:

- Since
$$\dot{x}_1 = x_1 - x_2^2,$$
we have
$$\dot{x}_1 > 0 \quad \text{when} \quad x_1 > x_2^2, \qquad \dot{x}_1 < 0 \quad \text{when} \quad x_1 < x_2^2.$$
So the flow points to the right on the right side of the parabola, and to the left on the left side.

- Since
$$\dot{x}_2 = x_2(1 - x_1),$$
we have:
$$\dot{x}_2 > 0 \quad \text{if} \quad \begin{cases} x_2 > 0, \ x_1 < 1, \\ x_2 < 0, \ x_1 > 1, \end{cases}$$
and
$$\dot{x}_2 < 0 \quad \text{if} \quad \begin{cases} x_2 > 0, \ x_1 > 1, \\ x_2 < 0, \ x_1 < 1. \end{cases}$$
Thus the flow points upward in the upper-left and lower-right regions relative to the line $x_1 = 1$, and downward in the upper-right and lower-left regions.

In summary:

- $(0, 0)$ is a source,
- $(1, 1)$ and $(1, -1)$ are saddles.

Workspace for Problem 2: Explain your reasoning/work here.

Partial Differential Equations: Problem 3 [10 points]:

Consider Poisson's equation in the infinite strip $0 \le y \le d$, $-\infty < x < \infty$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \delta(x - x_0)\,\delta(y - y_0)\,,$$

with the boundary conditions

$$u(x, 0) = 0\,, \qquad \frac{\partial u}{\partial y}(x, d) = 0\,,$$

where (x_0, y_0) is a point inside the strip, $0 < y_0 < d$.

Using the method of images, construct the Green's function $G(x, y; x_0, y_0)$ as an infinite series of image sources. Explain why a finite number of images cannot satisfy both boundary conditions, provide a sketch of the image system showing the locations and signs of the images, and verify that G satisfies the conditions on both walls. Comment on the convergence of the infinite series.

You may use the identity

$$\sum_{n=-\infty}^{+\infty} \ln\left[\alpha^2 + (\beta - nd)^2\right] = \ln\left[\cosh\left(\frac{\pi\alpha}{d}\right) - \cos\left(\frac{\pi\beta}{d}\right)\right] + \text{const.}$$

to express the result in closed form.

Workspace for Problem 3: Explain your reasoning/work here.

Solution

Free-space Green's function. The Green's function for the 2D Laplacian on the full plane satisfies $\nabla^2 G_\infty = \delta(\mathbf{x} - \boldsymbol{\xi})$ and is

$$G_\infty(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{2\pi} \ln r\,, \qquad r = |\mathbf{x} - \boldsymbol{\xi}| = \sqrt{(x - \xi_1)^2 + (y - \xi_2)^2}\,.$$

Key insight: mixed BCs change the image pattern.

The boundary conditions are: Dirichlet ($G = 0$) on $y = 0$ and Neumann ($\partial G/\partial y = 0$) on $y = d$.

- Reflection across a *Dirichlet* wall ($y = 0$): the image has *opposite* sign.
- Reflection across a *Neumann* wall ($y = d$): the image has the *same* sign.

This is the crucial difference from the Dirichlet–Dirichlet case. The sign pattern of the images is no longer a simple alternation $+, -, +, -, \ldots$

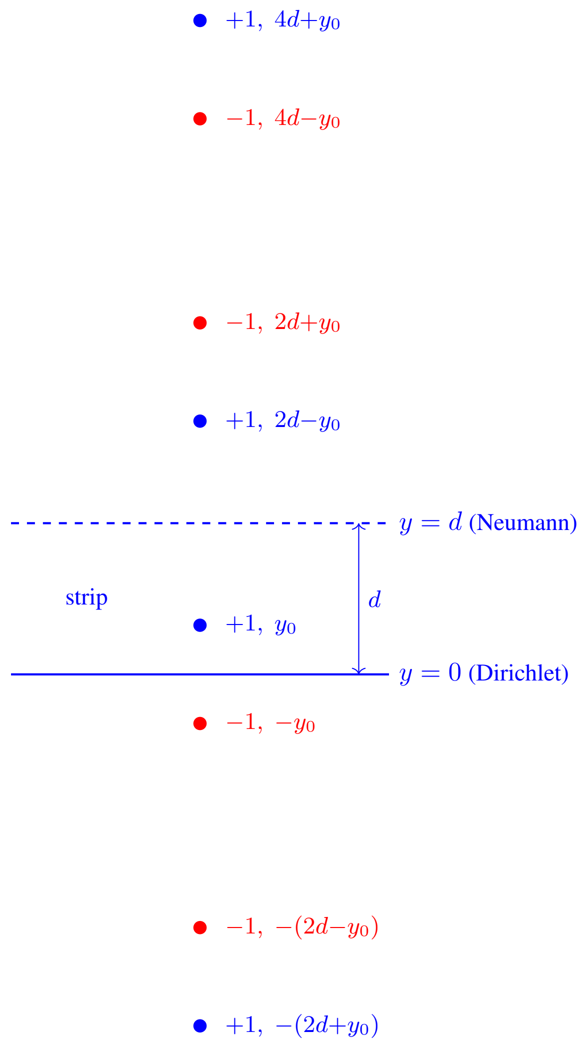
Building the image system.

Start with the source at (x_0, y_0) with sign $+1$.

1. Reflect across $y = 0$ (Dirichlet): image at $(x_0, -y_0)$ with sign -1.
2. Reflect both across $y = d$ (Neumann, same sign): source at $y_0 \to$ image at $2d - y_0$, sign $+1$; image at $-y_0 \to$ image at $2d + y_0$, sign -1.
3. Reflect those two new images across $y = 0$ (Dirichlet, flip sign): image at $2d - y_0 \to$ image at $-(2d - y_0)$, sign -1; image at $2d + y_0 \to$ image at $-(2d + y_0)$, sign $+1$.
4. Reflect across $y = d$ (Neumann, same sign): continues the pattern.

The period of the image system is $4d$ (not $2d$ as in the Dirichlet–Dirichlet case). The full set of images:

Position	Sign	Role	
$y_0 + 4nd$	$+1$	sources	
$-y_0 + 4nd$	-1	Dirichlet reflections of sources	$n \in \mathbb{Z}$
$2d - y_0 + 4nd$	$+1$	Neumann reflections of sources	
$2d + y_0 + 4nd$	-1	Dirichlet reflections of Neumann images	



The sign pattern within one period is: $+1, -1, +1, -1$ (at positions y_0, $-y_0$, $2d - y_0$, $2d + y_0$), repeating with period $4d$.

Why a finite number of images fails.

A single image at $-y_0$ enforces $G = 0$ on $y = 0$, but its Neumann reflection across $y = d$ creates a new source that violates $G = 0$ on $y = 0$. Each correction for one wall propagates a violation to the other. Only the infinite series achieves both conditions simultaneously.

Green's function (series form).

Grouping images by their y-positions:

$$G = \frac{1}{4\pi} \sum_{n=-\infty}^{+\infty} \Big[\ln\Big((x-x_0)^2 + (y - y_0 - 4nd)^2\Big) - \ln\Big((x-x_0)^2 + (y + y_0 - 4nd)^2\Big) + \ln\Big((x-x_0)^2 + (y - (2d-y_0) - 4nd)^2\Big) - \ln\Big((x-x_0)^2 + (y - (2d+y_0) - 4nd)^2\Big) \Big]. \tag{1}$$

Verification on $y = 0$ (Dirichlet):

Setting $y = 0$, the first and second ln terms become $\ln[\cdots + (y_0 + 4nd)^2]$ and $\ln[\cdots + (y_0 - 4nd)^2]$; pairing $+n$ with $-n$ shows they cancel. Similarly, the third and fourth terms become $\ln[\cdots + (2d - y_0 + 4nd)^2]$ and $\ln[\cdots + (2d + y_0 + 4nd)^2]$; reindexing $m = -n$ in one and $m = n$ in the other shows cancellation. Hence $G(x, 0; x_0, y_0) = 0$. ✓

Verification on $y = d$ (Neumann):

We need $\partial G/\partial y = 0$ at $y = d$. Differentiating each $\ln$ term produces expressions of the form $\frac{2(y-Y_n)}{(x-x_0)^2+(y-Y_n)^2}$. At $y = d$, the four image positions Y_n within one period (relative to $y = d$) are $d - y_0 - 4nd$, $d + y_0 - 4nd$, $y_0 - d - 4nd$, $-y_0 - d - 4nd$. The first and third are negatives of each other (at $n \leftrightarrow -n$), contributing derivatives that cancel in pairs due to the matching signs. Similarly for the second and fourth. Hence $\partial G/\partial y|_{y=d} = 0$. ✓

Closed-form summation.

Each of the four sums in (1) is of the form $\sum_n \ln[\alpha^2 + (\beta - 4nd)^2]$. Applying the given identity with d replaced by $4d$ (i.e., the period of the image system):

$$S_1 = \sum_n \ln\left[\alpha^2 + (y - y_0 - 4nd)^2\right] = \ln\left[\cosh\frac{\pi\alpha}{4d} - \cos\frac{\pi(y - y_0)}{4d}\right] + C\,,$$

$$S_2 = \sum_n \ln\left[\alpha^2 + (y + y_0 - 4nd)^2\right] = \ln\left[\cosh\frac{\pi\alpha}{4d} - \cos\frac{\pi(y + y_0)}{4d}\right] + C\,,$$

$$S_3 = \sum_n \ln\left[\alpha^2 + (y - 2d + y_0 - 4nd)^2\right] = \ln\left[\cosh\frac{\pi\alpha}{4d} - \cos\frac{\pi(y - 2d + y_0)}{4d}\right] + C\,,$$

$$S_4 = \sum_n \ln\left[\alpha^2 + (y - 2d - y_0 - 4nd)^2\right] = \ln\left[\cosh\frac{\pi\alpha}{4d} - \cos\frac{\pi(y - 2d - y_0)}{4d}\right] + C\,,$$

where $\alpha = x - x_0$ and the constant C is the same for all four (depends only on α and d), so it cancels in $G = \frac{1}{4\pi}(S_1 - S_2 + S_3 - S_4)$:

$$\boxed{G = \frac{1}{4\pi}\ln\left[\frac{\left(\cosh\frac{\pi\alpha}{4d} - \cos\frac{\pi(y-y_0)}{4d}\right)\left(\cosh\frac{\pi\alpha}{4d} - \cos\frac{\pi(y-2d+y_0)}{4d}\right)}{\left(\cosh\frac{\pi\alpha}{4d} - \cos\frac{\pi(y+y_0)}{4d}\right)\left(\cosh\frac{\pi\alpha}{4d} - \cos\frac{\pi(y-2d-y_0)}{4d}\right)}\right], \quad \alpha = x - x_0\,.}$$

Comment on convergence.

Each individual $\ln r_n$ diverges as $|n| \to \infty$ (logarithmic growth), so the series of individual source contributions diverges. However, within each period the *net charge* (sum of signs) is $+1 - 1 + 1 - 1 = 0$, so the images form a charge-neutral unit cell. The potential of a neutral collection of sources decays as $1/r^2$ (dipolar) or faster at large distances, ensuring that the *grouped* sum converges. This is why the series must be written with the numerator and denominator $\ln$ terms paired together (as a ratio inside the logarithm), not as separate divergent sums. The closed-form expression confirms the convergence: each $\cosh - \cos$ factor is finite and positive.

Workspace for Problem 3: Explain your reasoning/work here.

Workspace for Problem 3: Explain your reasoning/work here.

Partial Differential Equations: Problem 4 [10 points]:

Consider the wave equation on a finite domain $0 \le x \le 1$, $t \ge 0$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\,, \qquad u(x,0) = 0\,, \quad u_t(x,0) = 0\,,$$

with the boundary conditions

$$u(0,t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}\,, \qquad u(1,t) = 0\,.$$

Using the Laplace transform in time, find $u(x,t)$. Determine the exact time t^* at which the wavefront reaches the midpoint $x = 1/2$ after reflecting from $x = 1$.

Sketch u as a function of x at a time before and after t^*.

A short table of Laplace Transforms is provided.

Some Properties of the Laplace Transform		
Operation or function	$f(t)$	$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt$
First derivative	$\frac{df}{dt}$	$s\bar{f}(s) - f(0)$
Second derivative	$\frac{d^2 f}{dt^2}$	$s^2\bar{f}(s) - sf(0) - f'(0)$
Time scaling	$f(at)$	$\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$
Time shift	$f(t-a)H(t-a)$	$e^{-as}\bar{f}(s)$
Frequency shift	$e^{at}f(t)$	$\bar{f}(s-a)$
Dirac delta function	$\delta(t-a)$	e^{-as}
Heaviside function	$H(t-a)$	$\frac{e^{-as}}{s}$
One	1	$\frac{1}{s}$
Integer power	$t^n,\ n = 0,1,2\ldots$	$\frac{n!}{s^{n+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin(at)$	$\frac{a}{s^2+a^2}$, $\Re(s) > \lvert\Im(a)\rvert$
Cosine	$\cos(at)$	$\frac{s}{s^2+a^2}$, $\Re(s) > \lvert\Im(a)\rvert$
Geometric series	$\frac{1}{1-\epsilon} = \sum_{n=0}^{\infty} \epsilon^n$	$\lvert\epsilon\rvert < 1$

Workspace for Problem 4: Explain your reasoning/work here.

Solution

Step 1: Rewrite the BC.

$$u(0,t) = t - (t-1)\,H(t-1)\,.$$

Check: for $t < 1$ gives t; for $t > 1$ gives $t - (t-1) = 1$. ✓

Step 2: Laplace transform. With zero ICs, the transformed PDE is

$$s^2\bar{u} = c^2\bar{u}_{xx} \implies \bar{u}_{xx} - \frac{s^2}{c^2}\bar{u} = 0\,.$$

General solution: $\bar{u}(x,s) = A\,e^{-sx/c} + B\,e^{+sx/c}$.

Unlike the semi-infinite case, both exponentials must be retained.

Step 3: Boundary conditions in transform space.

At $x = 0$:

$$\bar{u}(0,s) = A + B = \frac{1-e^{-s}}{s^2}\,.$$

At $x = 1$:

$$\bar{u}(1,s) = Ae^{-s/c} + Be^{s/c} = 0\,.$$

From the second equation: $A = -Be^{2s/c}$. Substituting into the first:

$$-Be^{2s/c} + B = \frac{1-e^{-s}}{s^2} \implies B(1-e^{2s/c}) = \frac{1-e^{-s}}{s^2} \implies B = -\frac{1-e^{-s}}{s^2(e^{2s/c}-1)}\,,$$

$$A = \frac{(1-e^{-s})\,e^{2s/c}}{s^2(e^{2s/c}-1)} = \frac{1-e^{-s}}{s^2(1-e^{-2s/c})}\,.$$

Therefore:

$$\bar{u}(x,s) = \frac{1-e^{-s}}{s^2}\cdot\frac{e^{-sx/c} - e^{-s(2-x)/c}}{1-e^{-2s/c}}\,.$$

Step 4: Geometric series expansion.

Using $\frac{1}{1-e^{-2s/c}} = \sum_{n=0}^{\infty} e^{-2ns/c}$ for $\mathrm{Re}(s) > 0$:

$$\bar{u}(x,s) = \frac{1-e^{-s}}{s^2}\sum_{n=0}^{\infty}\left[e^{-s(x+2n)/c} - e^{-s(2-x+2n)/c}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{e^{-s\tau_n^+}}{s^2} - \frac{e^{-s\tau_n^-}}{s^2} - \frac{e^{-s(1+\tau_n^+)}}{s^2} + \frac{e^{-s(1+\tau_n^-)}}{s^2}\right],$$

where we define

$$\tau_n^+ = \frac{x+2n}{c}\,, \qquad \tau_n^- = \frac{2-x+2n}{c}\,.$$

Step 5: Inversion.

Each term has the form e^{-as}/s^2 with $\mathcal{L}^{-1}\{e^{-as}/s^2\} = (t-a)\,H(t-a)$.

$$\boxed{u(x,t) = \sum_{n=0}^{\infty}\Big[(t-\tau_n^+)H(t-\tau_n^+) - (t-\tau_n^-)H(t-\tau_n^-) - (t-1-\tau_n^+)H(t-1-\tau_n^+) + (t-1-\tau_n^-)H(t-1-\tau_n^-)\Big].}$$

At any finite time, only finitely many Heaviside terms are active.

Step 6: Time of first reflection reaching the midpoint.

The wavefront leaves $x = 0$ at $t = 0$ and travels at speed c.

- It first reaches the wall at $x = 1$ at time $t = 1/c$.
- It reflects (with sign change, due to the Dirichlet condition $u(1,t) = 0$) and travels back.
- It reaches the midpoint $x = 1/2$ at time $t = 1/c + (1/2)/c = 3/(2c)$.

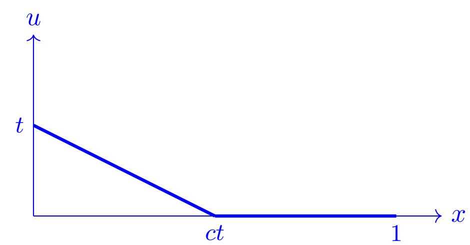
$$\boxed{t^* = \frac{3}{2c}.}$$

This corresponds to the activation of the $n = 0$ second term ($\tau_0^- = (2 - x)/c$) at $x = 1/2$: $\tau_0^- = (2 - 1/2)/c = 3/(2c) = t^*$. ✓

Step 7: Sketches.

Before t^* (e.g., $t = 1/(2c)$, before any reflection):

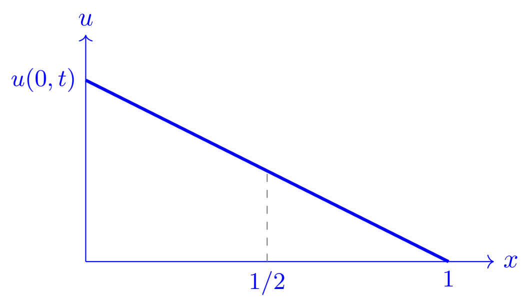
Only the $n = 0$ first term contributes: $u = (t - x/c)\, H(t - x/c)$. This is a ramp from $u(0,t) = t$ down to $u = 0$ at $x = ct$.



$t < 1/c$, no reflection yet

After t^* (e.g., $t = 2/c$, after first reflection):

Two terms active for $n = 0$: the direct ramp and the reflected (negative) ramp from $x = 1$. The reflected wave subtracts from the direct wave, creating a kink.



$t > t^*$, reflection present

As $t \to \infty$, the bouncing ramps accumulate and the solution approaches the steady state $u_{ss}(x) = 1 - x$ (the solution to $u_{xx} = 0$ with $u(0) = 1$, $u(1) = 0$).

Verification:

- $u(x, 0) = 0$: all Heavisides vanish at $t = 0$. ✓
- $u_t(x, 0) = 0$: time derivatives of each term involve Heavisides that vanish at $t = 0$ for $x > 0$. ✓
- $u(1,t) = 0$: at $x = 1$, $\tau_n^+ = (1 + 2n)/c$ and $\tau_n^- = (1 + 2n)/c$, so the terms cancel pairwise. ✓
- $u(0,t) = t - (t-1)H(t-1)$: at $x = 0$, $\tau_n^+ = 2n/c$ and $\tau_n^- = (2 + 2n)/c = \tau_{n+1}^+$, so consecutive terms telescope, leaving only the first: $u(0,t) = t - (t-1)H(t-1)$. ✓
- Each term is of the form $g(t \pm x/c)$, satisfying $u_{tt} = c^2 u_{xx}$. ✓