MAE Preliminary Examination

Mathematics Section

Monday, November 17, 2025, 1:00pm-3:30pm

Your Name

THREE OF THE FOUR PROBLEMS WILL BE GRADED **Select the 3 problems you have worked on, to be graded:**	**points**
Problem:	/10
Problem:	/10
Problem:	/10
Total	/30

Please give your answers/work in the space provided

Explain your work/steps clearly

Linear Algebra: Problem 1 [10 points]:

Let $A \in \mathbb{R}^{m\times n}$. We say an $n \times m$ matrix L is a *left inverse* of A if $LA = I_n$. Likewise, we say an $n \times m$ matrix R is a *right inverse* of A if $AR = I_m$. Do the following:

(a) Show that A has a *left inverse* if and only if the columns of A are linearly independent.

(b) Show that A has a *right inverse* if and only if the rows of A are linearly independent.

(c) If A has both a left and a right inverse, deduce that A must be square, invertible, and that the left and right inverses coincide, i.e., $L = R$.

Workspace for Problem 1: Explain your reasoning/work here.

Solution

a) Suppose $LA = I_n$. If $\sum_{j=1}^{n} c_j a_j = 0$ (with a_j the columns of A), then

$$0 = L\Big(\sum_{j=1}^{n} c_j a_j\Big) = \sum_{j=1}^{n} c_j (La_j) = \sum_{j=1}^{n} c_j e_j = (c_1, \dots, c_n)^\top,$$

so all $c_j = 0$. Hence the columns of A are linearly independent, and $\mathrm{rank}(A) = n$. On the other hand, if $\mathrm{rank}(A) = n$, then A has full column rank, so $A^\top A$ is $n \times n$ symmetric positive definite and thus invertible. Define $L := (A^\top A)^{-1} A^\top$. Then $LA = (A^\top A)^{-1} A^\top A = I_n$. Thus, L is a left inverse.

b) Suppose $AR = I_m$. If $x^\top A = 0$ for some $x \in \mathbb{R}^m$, then

$$x^\top = x^\top I_m = x^\top (AR) = (x^\top A)R = 0,$$

so $x = 0$. Thus the rows of A are linearly independent and $\mathrm{rank}(A) = m$. On the other hand, if $\mathrm{rank}(A) = m$, then $AA^\top$ is $m \times m$ symmetric positive definite and invertible. Define $R := A^\top (AA^\top)^{-1}$. Then $AR = A\,A^\top (AA^\top)^{-1} = I_m$. Thus, R is a right inverse.

c) If $LA = I_n$ and $AR = I_m$, then

$$n = \mathrm{rank}(I_n) = \mathrm{rank}(LA) \le \mathrm{rank}(A) \le \min\{m, n\} \implies n \le m,$$

$$m = \mathrm{rank}(I_m) = \mathrm{rank}(AR) \le \mathrm{rank}(A) \le \min\{m, n\} \implies m \le n.$$

Hence $m = n$ and $\mathrm{rank}(A) = n$, i.e. A is square and full rank, thus invertible. Finally,

$$L = LI_n = L(AR) = (LA)R = I_n R = R,$$

and since A is invertible, the (two-sided) inverse is unique: $L = R = A^{-1}$.

Workspace for Problem 1: Explain your reasoning/work here.

Differential Equations: Problem 2 [10 points]:

Consider $M(t) \in \mathbb{R}^{n\times n}$, for $t \in [0,1]$, that satisfies the differential equation

$$\frac{d}{dt}M(t) = M(t)M(t)^\top - M(t)^\top M(t), \text{ with initial condition } M(0) = M_0.$$

Here, $^\top$ denotes transposition. Do the following:

i) Show that the skew-symmetric part of $M(t)$, given by $X(t) = \frac{1}{2}(M(t) - M(t)^\top)$, remains constant with time, i.e., $X(t) = X(0) \in \mathbb{R}^{n\times n}$ for any t.

ii) Show that $X(0)$ is indeed a skew-symmetric matrix, i.e., that it satisfies $X(0) = -X(0)^\top$. Deduce that the matrix exponential $U(t) = e^{X(0)t}$ is an orthogonal matrix for any t.

iii) Now, consider the symmetric part of $M(t)$, given by $Y(t) = \frac{1}{2}(M(t) + M(t)^\top)$. Determine $Y(t)$ as a function of $Y(0)$ and $X(0)$. Hint: Write down a differential equation for the symmetric part $Y(t)$, and solve it by using an suitable change of variables.

iv) Deduce $M(t)$ as a function of $X(0)$ and $Y(0)$, i.e., the skew-symmetric and symmetric parts of $M(0)$ as above.

Workspace for Problem 2: Explain your reasoning/work here.

Solution

i) Taking the time-derivative of $X(t)$ gives

$$\begin{aligned}\frac{d}{dt}X(t) &= \frac{1}{2}\frac{d}{dt}(M(t) - M(t)^\top)\\ &= \frac{1}{2}(M(t)M(t)^\top - M(t)^\top M(t) - (M(t)M(t)^\top - M(t)^\top M(t))) = 0.\end{aligned}$$

Thus, $X(t)$ remains constant with time and equals the initial value $X(0)$.

ii) Skew-symmetry of $X(t)$ follows by construction, since

$$X(0)^\top = \frac{1}{2}(M(t) - M(t)^\top)^\top = \frac{1}{2}(M(t)^\top - M(t)) = -\frac{1}{2}(M(t) - M(t)^\top) = -X(0).$$

Now, let $U(t) = e^{X(0)t}$. Using the skew-symmetry of $X(0)$, we have for any t

$$U(t)^\top U(t) = U(t)U(t)^\top = e^{X(0)t}e^{X(0)^\top t} = e^{X(0)t}e^{-X(0)t} = e^{(X(0)-X(0))t} = I.$$

Thus, $U(t)$ is indeed an orthogonal matrix for any t.

iii) Note that $M(t) = X(t) + Y(t) = X(0) + Y(t)$, $X(0)^\top = -X(0)$, and $Y(t)^\top = Y(t)$. Plugging back into the equation for $M(t)$ gives

$$\begin{aligned}\frac{d}{dt}Y(t) &= (X(0) + Y(t))(X(0) + Y(t))^\top - (X(0) + Y(t))^\top(X(0) + Y(t))\\ &= 2(X(0)Y(t) - Y(t)X(0)).\end{aligned}$$

Now, consider the change of variables $Z(t) = e^{-2X(0)t}Y(t)e^{2X(0)t}$, $Z(0) = Y(0)$. Then,

$$\frac{d}{dt}Z(t) = e^{2X(0)t}\left(\frac{d}{dt}Y(t) - 2(X(0)Y(t) - Y(t)X(0))\right)e^{-2X(0)t} = 0.$$

Thus, $Z(t) = Z(0) = Y(0)$ for all t, from which we obtain

$$Z(0) = e^{-2X(0)t}Y(t)e^{2X(0)t} \implies Y(t) = e^{2X(0)t}Y(0)e^{-2X(0)t}, \text{ as desired.}$$

iv) Having found $Y(t)$ as a function of $X(0)$ and $Y(0)$, $M(t)$ is simply $M(t) = X(0) + Y(t) = X(0) + e^{2X(0)t}Y(0)e^{-2X(0)t}$.

Workspace for Problem 2: Explain your reasoning/work here.

Partial Differential Equations: Problem 3 [10 points]:

Consider the wave equation for $u(t,x)$ in one spatial dimension for semi-infinite space and time, $x \geq 0, t \geq 0$.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} u(0,x) &= 0 \\ \frac{\partial u}{\partial t}(0,x) &= 0 \\ u(t,0) &= 1\ ,\ 0 \leq t \leq 1 \\ u(t,0) &= 0\ , t > 1 \end{aligned}$$

Solve by two methods: (i) d'Alembert's method and (ii) the Laplace transform applied over time. Compare the two solutions. Sketch the solution on the $x-t$ plane. A short table of Laplace Transforms is provided.

Some Properties of the Laplace Transform		
Operation or function	$f(t)$	$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt$
First derivative	$\frac{df}{dt}$	$s\bar{f}(s) - f(0)$
Second derivative	$\frac{d^2 f}{dt^2}$	$s^2\bar{f}(s) - sf(0) - f'(0)$
Time scaling	$f(at)$	$\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$
Time shift	$f(t-a)H(t-a)$	$e^{-as}\bar{f}(s)$
Frequency shift	$e^{at}f(t)$	$\bar{f}(s-a)$
Dirac delta function	$\delta(t-a)$	e^{-as}
Heaviside function	$H(t-a)$	$\frac{e^{-as}}{s}$
One	1	$\frac{1}{s}$
Integer power	$t^n,\ n = 0, 1, 2 \ldots$	$\frac{n!}{s^{n+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin(at)$	$\frac{a}{s^2+a^2}$, $\Re(s) > \lvert\Im(a)\rvert$
Cosine	$\cos(at)$	$\frac{s}{s^2+a^2}$, $\Re(s) > \lvert\Im(a)\rvert$

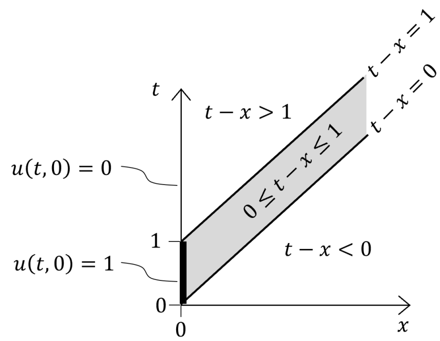
Workspace for Problem 3: Explain your reasoning/work here.

Solution

d'Alembert Solution. State $\xi = x - t, \eta = x + t$. Transform the PDE to obtain a new PDE and find two simple general solutions in terms of yet incompletely defined functions which may be added.

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0\ ;\ u(\xi, \eta) = f(\xi) + g(\eta);\ u(t, x) = f(x - t) + g(x + t)$$

Identifying the lines $\xi = x - t$ and $\eta = x + t$ as the characteristics, we sketch the regimes of the solution on the $x - t$ diagram, which helps in visualizing and developing the solution. It is evident that there is no boundary condition that can generate characteristics of the type $\eta = x + t$, therefore we expect that the solution will contain only $u(t, x) = f(x - t)$. However, we keep the g solution for now and let the math determine whether g exists or not.



Recalling that $x \geq 0, t \geq 0$, it is important to note that the argument $\xi = x - t$ of the function f is positive for $t - x < 0$ and negative for $t - x > 0$; whereas the argument $\eta = x + t$ of the function g is non-negative everywhere.

Now we apply the initial and boundary conditions in the regimes identified on the sketch:

$t - x < 0$:

The initial conditions give:

$$u(0, x) = f(x) + g(x) = 0$$

$$\frac{\partial u}{\partial t}(0, x) = -f'(x) + g'(x) = 0 \rightarrow -f(x) + g(x) = C$$

where C is constant of integration. Adding and subtracting we obtain $f = -C/2$ and $g = C/2$. Because $x \geq 0$, this means that g (whose argument is always non-negative) is now known throughout our domain: $g(\eta) = C/2$ everywhere. However, f is known only for non-negative argument, so we state $f(\xi) = -C/2$ only within the region considered here. The full solution is $u = f + g = 0$ in this region

$0 \leq t - x \leq 1$:

The boundary condition requires

$$u(t, 0) = f(-t) + g(t) = 1$$

We already know $g = C/2$, so $f = 1 - C/2$. The full solution is $u = f + g = 1$ in this region.

$t - x > 1$:

The boundary condition requires

$$u(t, 0) = f(-t) + g(t) = 0$$

Using $g = C/2$, $f = -C/2$. The full solution is $u = f + g = 0$ in this region.

We conclude,

$$u(t, x) = \begin{cases} 0\,, & t - x < 0 \\ 1\,, & 0 \leq t - x \leq 1 \\ 0\,, & t - x > 1 \end{cases}$$

It is evident that the constant C is immaterial to the solution and could have been selected as $C = 0$ (thus setting $g = 0$ everywhere, consistent with our expectation) without loss of completeness.

Laplace Transform Solution. Transform the PDE and boundary condition in time to obtain the ODE and BC. The Laplace-transformed solution is denoted $\bar{u}(s,x)$. Applying the rule for the Laplace transform of the second derivative,

$$\frac{d^2\bar{u}}{dx^2} = s^2\bar{u} - su(0,x) - \frac{\partial u}{\partial t}(0,x)$$

The initial conditions force the second and third terms on the right-hand side to vanish, resulting in the ODE

$$\frac{d^2\bar{u}}{dx^2} - s^2\bar{u} \;=\; 0$$

The transformed BC is

$$\bar{u}(s,0) = \int_0^\infty u(t,0)e^{-st}dt = \int_0^1 e^{-st}dt = \frac{1}{s} - \frac{e^{-s}}{s}$$

Two exponential homogeneous solutions are found: e^{-sx} and e^{sx}. Given that $\Re(s) > 0$, the latter is eliminated because it does not decay at infinity. Matching the BC,

$$\bar{u}(s,x) = \frac{e^{-sx}}{s} - \frac{e^{-s(x+1)}}{s}$$

Now we take the inverse transforms, recognizing that the first and second terms on the right-hand side correspond to the Heaviside function at time shifts x and $x+1$, respectively:

$$u(t,x) = H(t-x) \;-\; H(t-x-1)$$

Writing this more explicitly as

$$u(t,x) \;=\; \begin{cases} 0\,, & t-x<0 \\ 1\,, & 0 \le t-x \le 1 \\ 0\,, & t-x>1 \end{cases}$$

we recover the solution obtained using the d'Alembert approach.

Workspace for Problem 3: Explain your reasoning/work here.

Partial Differential Equations: Problem 4 [10 points]:

Consider Poisson's equation for $u(x, y)$ in semi-infinite space, $0 \le x \le \infty, 0 \le y \le \infty$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \;=\; f(x, y)$$

subject to the boundary conditions

$$u(x, 0) = 0$$

$$\frac{\partial u}{\partial x}(0, y) = g(y)$$

The function $f(x, y)$ takes the values

$$f(x, y) \;=\; \begin{cases} 1\,, & 1 \le x \le 2,\; 1 \le y \le 2 \\ 0\,, & x < 1,\; x > 2,\; y < 1,\; y > 2 \end{cases}$$

The function $g(y)$ takes the values

$$g(y) \;=\; \begin{cases} 1\,, & 1 \le y \le 2 \\ 0\,, & y < 1,\; y > 2 \end{cases}$$

Recalling Green's theorem for Poisson's equation in 2D semi-infinite space, with boundary data given on the x and y axes,

$$\begin{aligned} u(x, y) \;&=\; \int_0^\infty \int_0^\infty f(\xi, \eta)\, G(x, y; \xi, \eta) d\xi d\eta \;+ \\ &+\; \int_0^\infty \left[\frac{\partial u}{\partial \eta}(\xi, 0)\, G(x, y; \xi, 0) \;-\; u(\xi, 0)\, \frac{\partial G}{\partial \eta}(x, y; \xi, 0) \right] d\xi \\ &+\; \int_0^\infty \left[\frac{\partial u}{\partial \xi}(0, \eta)\, G(x, y; 0, \eta) \;-\; u(0, \eta)\, \frac{\partial G}{\partial \xi}(x, y; 0, \eta) \right] d\eta \end{aligned}$$

construct the Green's function $G(x, y; \xi, \eta)$ for this problem using the method of images. Then, apply the Green's function and present the solution of $u(x, y)$ as a sum of integrals with the integrands reduced to the simplest forms and the limits on the integrals simplified to give the smallest needed range of integration. Final integration is not required. Provide a sketch to clarify how the method of images is applied.

Workspace for Problem 4: Explain your reasoning/work here.

Solution

We first create a Green's function $G(x, y; \xi, \eta)$ for the self-adjoint partial differential operator (i.e., the Laplacian operation). The Green's function, following Green's theorem and the original defined problem, is governed by the following relations:

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \;=\; \delta(x - \xi, y - \eta) \;\; ; \;\; G(x, y, \xi, 0) = 0 \;\; ; \;\; \frac{\partial G}{\partial x}(x, y, 0, \eta) = 0$$

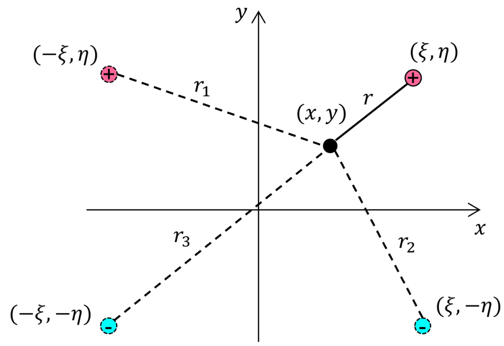
The solution process in the next step with Green's theorem yields after setting zero-valued boundary conditions for $u, \partial u/\partial x$, G and $\partial G/\partial x$ at $x = 0$ and $y = 0$:

$$\begin{aligned} u(x, y) = \quad & \int_0^\infty \int_0^\infty G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta \\ & + \int_0^\infty \left[G(x, y; 0, \eta) \frac{\partial u}{\partial \xi}(0, \eta) \;-\; u(0, \eta) \frac{\partial G}{\partial \xi}(x, y; 0, \eta) \right] d\eta \\ = \quad & \int_0^\infty \int_0^\infty G(x, y; \xi, \eta)\, f(\xi, \eta)\, d\xi d\eta + \int_0^\infty G(x, y; 0, \eta)\, g(\eta)\, d\eta \end{aligned}$$

Now, create the Green's function precisely. The basic solution for G becomes simple when we write the PDE in cylindrical coordinates centered on the singularity at (x, y) = (ξ, η):

$$\frac{1}{r} \frac{\partial}{\partial r} \Big(\frac{r \partial G}{\partial r} \Big) = \delta(r) \; ; \; r(x, y; \xi, \eta) \equiv \sqrt{(x - \xi)^2 + (y - \eta)^2} \; , \; \int 2\pi r \delta(r) dr = 1$$

Thus, two successive integrations yield $G = (1/2\pi)\ln(r)$. However, this basic form does not satisfy boundary conditions. So, an image source is added at $(-\xi, \eta)$ and image sinks are added at $(\xi, -\eta)$ and $(-\xi, -\eta)$. See the sketch for image locations.



The final form of G is

$$G = \frac{1}{2\pi}[\ln r + \ln r_1 - \ln r_2 - \ln r_3] = \frac{1}{2\pi}\ln\left[\frac{r r_1}{r_2 r_3}\right]$$

$$r_1(x, y; \xi, \eta) \equiv \sqrt{(x+\xi)^2 + (y-\eta)^2}$$
$$r_2(x, y; \xi, \eta) \equiv \sqrt{(x-\xi)^2 + (y+\eta)^2}$$
$$r_3(x, y; \xi, \eta) \equiv \sqrt{(x+\xi)^2 + (y+\eta)^2}$$

Finally, we reflect the fact that f and g are known with zero values over portions of the domain:

$$u(x, y) = \frac{1}{2\pi}\int_1^2\int_1^2 \ln\left[\frac{r\ r_1}{r_2\ r_3}\right] d\xi d\eta + \frac{1}{2\pi}\int_1^2 \ln\left[\frac{r\ r_1}{r_2\ r_3}\right]_{\xi=0} d\eta$$

Workspace for Problem 4: Explain your reasoning/work here.