MAE Preliminary Examination

Mathematics Section

Monday, April 17, 2023, 9:00am-11:30noon

Your Name

THREE PROBLEMS WILL BE GRADED **Select the 3 problems you've worked, to be graded:**	**points**
Problem:	$/10$
Problem:	$/10$
Problem:	$/10$
Total	$/30$

Please give your answers/work in the space provided

Explain your work/steps clearly

Linear Algebra: Problem 1 [10 points]:

i) Explain why real symmetric matrics have real eigenvalues.

ii) Explain why eigenvectors of real symmetric matrices that correspond to different eigenvalues are orthogonal.

iii) Assume that $M = AB$, with A, B real symmetric matrices and $A > 0$ (positive definite). Show that M has real eigenvalues.
Hint: Show that M is similar to a symmetric matrix.

iv) Assume that $M = ABC$, with A, B, C real symmetric matrices, $A > 0$ (positive definite), and B, C commute, i.e., $BC = CB$. Show that M has real eigenvalues.
Hint: Show that BC is symmetric.

v) Consider real symmetric matrices A, B. Prove that if they commute, i.e., if $AB = BA$, then there is an orthogonal transformation $A \to UAU^T$, $B \to UBU^T$, so that both UAU^T and UBU^T are diagonal. For simplicity you may assume that all eigenvalues of A are distinct.

Hint: Recall that there is always an orthogonal matrix U so that UAU^T is diagonal.

Workspace for Problem 1: Explain your reasoning/work here.

Solution:
i) If $Ax = \lambda x$, with x eigenvector corresponding to eigenvalue λ, then $x^T A = \lambda x^T$ as well, with ^{T} denoting transpose. Hence

$$x^T Ax = \lambda x^T x \Rightarrow \lambda = \frac{x^T Ax}{x^T x},$$

which is a real number.

ii) If $Ax_i = \lambda_i x_i$, $i \in \{1, 2\}$, with $\lambda_1 \neq \lambda_2$, then by substituting $x_1^T A = \lambda_1 x_1$,

$$x_1^T Ax_2 = \lambda_1 x_1^T x_2.$$

Then, also, substituting $Ax_2 = \lambda_2 x_2$,

$$x_1^T Ax_2 = \lambda_2 x_1^T x_2.$$

Thus, $(\lambda_1 - \lambda_2)x_1^T x_2 = 0$. Since $\lambda_1 - \lambda_2 \neq 0$, it follows that $x_1^T x_2 = 0$.

iii) If $A^{\frac{1}{2}}$ is the symmetric square root of B, then

$$A^{-\frac{1}{2}} M A^{\frac{1}{2}} = A^{\frac{1}{2}} B A^{\frac{1}{2}},$$

which is symmetric. Hence, the eigenvalues of M are the same as those of $A^{\frac{1}{2}} B A^{\frac{1}{2}}$, which are real since it symmetric.

iv) Clearly $(BC)^T = CB = BC$, and since BC is symmetric, we apply iii).

v) Assume U as in the hint, and let $D_A = UAU^T$ and $D_B = UBU^T$. By assumption D_A is diagonal and it remains to show that D_B is also diagonal. Since $AB = BA$, and $UU^T = U^T U$ is the identity matrix, then

$$D_A D_B = UAU^T UBU^T = UABU^T,$$

and similarly, $D_B D_A = UBAU^T$, hence $D_A D_B = D_B D_A$. But unless D_B is diagonal, assuming as stated in the statement that the eigenvalues of A (diagonal entries of D_A) are distinct, D_B cannot commute with D_A; if D_B is not diagonal, the off diagonal entries will be scaled differently depending on whether D_A is multiplied from the left or from the right.

Differential Equations: Problem 2 [10 points]:

Consider a heat exchanger with two pipes in contact running liquid in opposite directions (typical for heat exchangers). That is, the one (we call upper) runs from left to right, and the other (we call lower) runs from right to left. The upper runs hot liquid that along the length of this contact releases heat to the lower one and cools down. The lower one, starting from a much lower temperature on the right, runs to the left and absorbs heat, so that its temperature increases.

The length we consider for both is $L = 1$ [units of length]. The temperature of the upper is $u(x)$ and the temperature of the lower is $\ell(x)$, where $x \in [0, L] = [0, 1]$. We assume steady state, in that the heat exchanger has been working for a while and the temperature of each pipe no longer depends on time, only on position x.

That is $x = 0$ designates one end, and $x = 1$ designates the other. We assume that

$$u(0) = 100 \text{ in } {}^oC, \text{ while}$$
$$\ell(1) = 0 \text{ in } {}^oC.$$

As the fluid in the upper flows to the right, it releases heat to the lower, and assuming flow velocity $v = 1$ [units of length per time] we obtain that the reduction (resp. increase) in temperature of the upper (resp. lower) pipe, as a function of position $x \in [0, 1]$ obeys

$$\frac{du(x)}{dx} = \ell(x) - u(x)$$
$$\frac{d\ell(x)}{dx} = \ell(x) - u(x).$$

Determine $u(1)$ and $\ell(0)$.

The physical explanation is as follows:
As fluid in the upper pipe runs from left to right, it releases heat to the lower one, and therefore the temperature of the upper pipe as a function of x decreases from left to right. On the other hand the lower pipe (the fluid in it that is), absorbs heat and its temperature increases from right to left.

Workspace for Problem 2: Explain your reasoning/work here.

Solution:
Let $\xi(x) = \binom{u(x)}{\ell(x)}$. The differential equation for the pair is

$$\frac{d}{dx}\xi(x) = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \xi(x).$$

The state transition matrix for $x = 1$ is

$$\exp(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix},$$

since

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = 0.$$

(Recall that the matrix exponential is $\exp(A) = I + A + \frac{1}{2}A^2 + \ldots$, and in our case $A^2 = 0$.)
Thus, we need to solve the system of equations

$$\begin{bmatrix} u(1) \\ 0^o C \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 100^o C \\ \ell(0) \end{bmatrix}.$$

Thus, $u(1) = \ell(0)$ and $2\ell(0) = 100$. Therefore,

$$u(1) = \ell(0) = 50^o C.$$

Alternative solution:
From $\dot{u}(x) = \ell(x) - u(x)$ and $\dot{\ell}(x) = \ell(x) - u(x)$, we have that $\dot{u}(x) - \dot{\ell}(x) = 0$ by subtracting the two equations. Hence $u(x) - \ell(x) = c$ a constant, and therefore

$$\dot{u}(x) = \dot{\ell}(x) = c$$

giving

$$\begin{aligned} u(x) &= 100 + cx \\ \ell(x) &= \ell(0) + cx. \end{aligned}$$

It follows that

$$\begin{aligned} u(1) &= 100 + c \\ 0 &= \ell(0) + c, \end{aligned}$$

by evaluating at $x = 0$ and $x = 1$, respectively.

Since $\ell(0) = -c$, from $\dot{u}(x) = \ell(x) - u(x)$,

$$\dot{u}(x) = (-c + cx) - (100 + cx),$$

but also, from before, $\dot{u}(x) = c$. Therefore,

$$c = -c - 100 \quad \Rightarrow \quad c = -50 \quad \Rightarrow \quad u(1) = 50^o C \text{ and } \ell(0) = 50^o C.$$

Problem 3 [10 points]:

Consider the first-order PDE

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} - 2u = 0$$

for the domain $-\infty < x < \infty, \quad t \geq 0$ with the initial conditions:

$$u(0,x) = \begin{cases} \exp(x) & -\infty < x \leq 0 \\ 0 & 0 < x \leq 1 \\ 1 & 1 < x \leq 3 \\ 0 & 3 < x < \infty \end{cases}$$

Find $u(t, x)$ in the domain following the suggested path.

1. Let the function $f(u, t, x) =$ constant be equivalent to the solution $u(t, x)$ and show that

$$\frac{\partial u}{\partial t} = -\frac{\partial f}{\partial t} / \frac{\partial f}{\partial u}$$

and

$$\frac{\partial u}{\partial x} = -\frac{\partial f}{\partial x} / \frac{\partial f}{\partial u}$$

2. Consequently, show that

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} - 2u \frac{\partial f}{\partial u} = 0$$

3. Explain what is implied about the local direction described by the vector $(1, -1, -2u)$ at each solution point.
4. Show the relations between differential changes du, dt, and dx at each point on the solution surface $f =$ constant.
5. Obtain integral relations between u, t, and x that satisfy the initial conditions and the original PDE.

Workspace for Problem 3: Explain your reasoning/work here.

Solution:

Etc...

Problem 4 [10 points]:

We have the heat equation

$$u_t - \alpha(u_{xx} + u_{yy}) = 0$$

for the infinite domain

$$-\infty < x < \infty, \quad -\infty < y < \infty, \quad \text{and} \quad t > 0$$

with $u(0, x, y) = 1$ in the rectangle $-1 \leq x \leq 1$ and $0 \leq y \leq 1$; $u(0, x, y) = 0$ outside of that rectangle.

1. Using a multidimensional Fourier transform, find $u(t, x, y)$ for the infinite domain in space and semi-infinite domain in time.
2. Explain where the solution becomes an even or odd function of x and/or y.
3. Why can't the Fourier transform be used to resolve the temporal behavior?
4. Why do the shorter wavelength components decay faster in time?

Table 1: Some Useful Fourier Transforms

$f(x) = \mathcal{F}^{-1}\{F(\omega)\} = \int_{-\infty}^{+\infty} F(\omega)e^{i\omega x}d\omega$	$\mathcal{F}\{f(x)\} = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx$
$f^n(x)$	$(i\omega)^n F(\omega)$
$f(x \pm a)$	$e^{\pm ia\omega}F(\omega)$
$f(x) * g(x) = \int_{-\infty}^{\infty} f(\bar{x})g(x - \bar{x})d\bar{x}$	$F(\omega)G(\omega)$
$e^{-a^2x^2}$	$\frac{1}{2a\sqrt{\pi}}e^{-\omega^2/4a^2}$
$e^{-a\|x\|}$	$\frac{1}{2\pi}\frac{2a}{a^2+\omega^2}$
$\frac{a}{a^2+\omega^2}$	$\frac{1}{2}e^{-a\|x\|}$
$\sin \omega_0 x$	$\frac{i}{2}[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
$\cos \omega_0 x$	$\frac{1}{2}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
$H(x)$	$\frac{1}{2\pi}\frac{1}{i\omega}$

Workspace for Problem 3: Explain your reasoning/work here.

Solution:

Etc...