Tribute to Rudolf E. Kalman

quest for algebraic characterizations of positivity

& personal reminiscences

Tryphon Georgiou
Univ. of California, Irvine

IFAC, Toulouse, July 2017

A tribute to

Rudolf E. Kalman

May 19, 1930 - July 2, 2016

The early (miracle) years

A new approach…
20,000+ citations

Kalman, Kalman filtering
~ 1,000,000 citations

A New Approach to Linear Filtering and Prediction Problems[1]

R. E. KALMAN
Research Institute for Advanced Study,[2]
Baltimore, Md.

The classical filtering and prediction problem is re-examined using the Bode-Shannon representation of random processes and the "state transition" method of analysis of dynamic systems. New results are:

(1) The formulation and methods of solution of the problem apply without modification to stationary and nonstationary statistics and to growing-memory and infinite-memory filters.

(2) A nonlinear difference (or differential) equation is derived for the covariance matrix of the optimal estimation error. From the solution of this equation the coefficients of the difference (or differential) equation of the optimal linear filter are obtained without further calculations.

(3) The filtering problem is shown to be the dual of the noise-free regulator problem.

The new method developed here is applied to two well-known problems, confirming and extending earlier results.

The discussion is largely self-contained and proceeds from first principles; basic concepts of the theory of random processes are reviewed in the Appendix.

Introduction

An important class of theoretical and practical problems in communication and control is of a statistical nature. Such problems are: (i) Prediction of random signals; (ii) separation of random signals from random noise; (iii) detection of signals of known form (pulses, sinusoids) in the presence of random noise.

In his pioneering work, Wiener [1][3] showed that problems (i) and (ii) lead to the so-called Wiener-Hopf integral equation; he also gave a method (spectral factorization) for the solution of this integral equation in the practically important special case of stationary statistics and rational spectra.

Many extensions and generalizations followed Wiener's basic work. Zadeh and Ragazzini solved the finite-memory case [2]. Concurrently and independently of Bode and Shannon [3], they also gave a simplified method [2] of solution. Booton discussed the nonstationary Wiener-Hopf equation [4]. These results are now in standard texts [5-6]. A somewhat different approach along these main lines has been given recently by Darlington [7]. For extensions to sampled signals, see, e.g., Franklin [8], Lees [9]. Another approach based on the eigenfunctions of the Wiener-Hopf equation (which applies also to nonstationary problems whereas the preceding methods in general don't), has been pioneered by Davis [10] and applied by many others, e.g., Shinbrot [11], Blum [12], Pugachev [13], Solodovnikov [14].

In all these works, the objective is to obtain the specification of a linear dynamic system (Wiener filter) which accomplishes the prediction, separation, or detection of a random signal.[4]

Present methods for solving the Wiener problem are subject to a number of limitations which seriously curtail their practical usefulness:

(1) The optimal filter is specified by its impulse response. It is not a simple task to synthesize the filter from such data.

(2) Numerical determination of the optimal impulse response is often quite involved and poorly suited to machine computation. The situation gets rapidly worse with increasing complexity of the problem.

(3) Important generalizations (e.g., growing-memory filters, nonstationary prediction) require new derivations, frequently of considerable difficulty to the nonspecialist.

(4) The mathematics of the derivations are not transparent. Fundamental assumptions and their consequences tend to be obscured.

This paper introduces a new look at this whole assemblage of problems, sidestepping the difficulties just mentioned. The following are the highlights of the paper:

(5) *Optimal Estimates and Orthogonal Projections.* The Wiener problem is approached from the point of view of conditional distributions and expectations. In this way, basic facts of the Wiener theory are quickly obtained; the scope of the results and the fundamental assumptions appear clearly. It is seen that all statistical calculations and results are based on first and second order averages; no other statistical data are needed. Thus difficulty (4) is eliminated. This method is well known in probability theory (see pp. 75–78 and 148–155 of Doob [15] and pp. 455–464 of Loève [16]) but has not yet been used extensively in engineering.

(6) *Models for Random Processes.* Following, in particular, Bode and Shannon [3], arbitrary random signals are represented (up to second order average statistical properties) as the output of a linear dynamic system excited by independent or uncorrelated random signals ("white noise"). This is a standard trick in the engineering applications of the Wiener theory [2–7]. The approach taken here differs from the conventional one only in the way in which linear dynamic systems are described. We shall emphasize the concepts of *state* and *state transition;* in other words, linear systems will be specified by systems of first-order difference (or differential) equations. This point of view is

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[1] This research was supported in part by the U. S. Air Force Office of Scientific Research under Contract AF 49 (638)-382.

[2] 7212 Bellona Ave.

[3] Numbers in brackets designate References at end of paper.

[4] Of course, in general these tasks may be done better by nonlinear filters. At present, however, little or nothing is known about how to obtain (both theoretically and practically) these nonlinear filters.

Contributed by the Instruments and Regulators Division and presented at the Instruments and Regulators Conference, March 29– April 2, 1959, of The American Society of Mechanical Engineers.

 Manuscript received at ASME Headquarters, February 24, 1959. Paper No. 59—IRD-11.

The early (miracle) years

The Theory of Optimal Control and the Calculus of Variations†

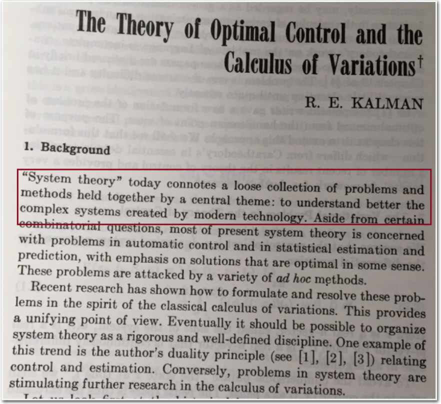
R. E. KALMAN

1. Background

"System theory" today connotes a loose collection of problems and methods held together by a central theme: to understand better the complex systems created by modern technology. Aside from certain combinatorial questions, most of present system theory is concerned with problems in automatic control and in statistical estimation and prediction, with emphasis on solutions that are optimal in some sense. These problems are attacked by a variety of *ad hoc* methods.

Recent research has shown how to formulate and resolve these problems in the spirit of the classical calculus of variations. This provides a unifying point of view. Eventually it should be possible to organize system theory as a rigorous and well-defined discipline. One example of this trend is the author's duality principle (see [1], [2], [3]) relating control and estimation. Conversely, problems in system theory are stimulating further research in the calculus of variations.

The early (miracle) years



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The problem is quite broad, and there are many unsettled questions. This

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Algebraic Theory of Linear Systems*

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Elementary textbooks in system theory present the Laplace transform as a magic and mysterious gadget. Somehow the difficult problem (?) of solving linear differential equations with constant coefficients is replaced with the easy problem (?) of factoring polynomials and computing partial fractions.

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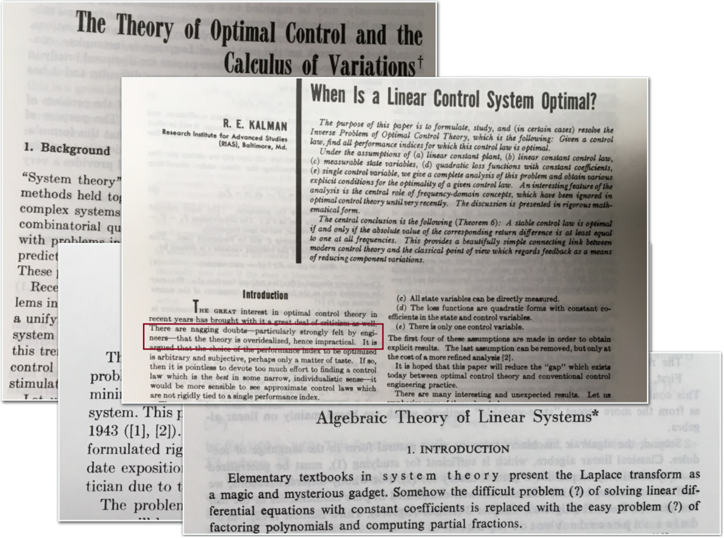
The Theory of Optimal Control and the Calculus of Variations†

1. Background

"System theory" methods held to complex systems combinatorial qu with problems in predict These Rece lems in a unify system this tre control stimulat

When Is a Linear Control System Optimal?

R. E. KALMAN
Research Institute for Advanced Studies (RIAS), Baltimore, Md.

The purpose of this paper is to formulate, study, and (in certain cases) resolve the Inverse Problem of Optimal Control Theory, which is the following: Given a control law, find all performance indices for which this control law is optimal.

Under the assumptions of (a) linear constant plant, (b) linear constant control law, (c) measurable state variables, (d) quadratic loss functions with constant coefficients, (e) single control variable, we give a complete analysis of this problem and obtain various explicit conditions for the optimality of a given control law. An interesting feature of the analysis is the central role of frequency-domain concepts, which have been ignored in optimal control theory until very recently. The discussion is presented in rigorous mathematical form.

The central conclusion is the following (Theorem 6): A stable control law is optimal if and only if the absolute value of the corresponding return difference is at least equal to one at all frequencies. This provides a beautifully simple connecting link between modern control theory and the classical point of view which regards feedback as a means of reducing component variations.

Introduction

THE GREAT interest in optimal control theory in recent years has brought with it a great deal of criticism as well. There are nagging doubts—particularly strongly felt by engineers—that the theory is overidealized, hence impractical. It is argued that the choice of the performance index to be optimized is arbitrary and subjective, perhaps only a matter of taste. If so, then it is pointless to devote too much effort to finding a control law which is the *best* in some narrow, individualistic sense—it would be more sensible to see approximate control laws which are not rigidly tied to a single performance index.

(c) All state variables can be directly measured.
(d) The loss functions are quadratic forms with constant coefficients in the state and control variables.
(e) There is only one control variable.

The first four of these assumptions are made in order to obtain explicit results. The last assumption can be removed, but only at the cost of a more refined analysis [2].

It is hoped that this paper will reduce the "gap" which exists today between optimal control theory and conventional control engineering practice.

There are many interesting and unexpected results. Let us

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Algebraic Theory of Linear Systems*

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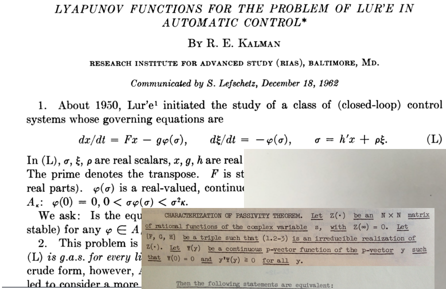


*LYAPUNOV FUNCTIONS FOR THE PROBLEM OF LUR'E IN AUTOMATIC CONTROL**

BY R. E. KALMAN

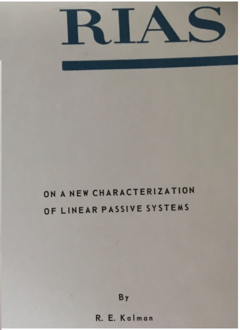
RESEARCH INSTITUTE FOR ADVANCED STUDY (RIAS), BALTIMORE, MD.

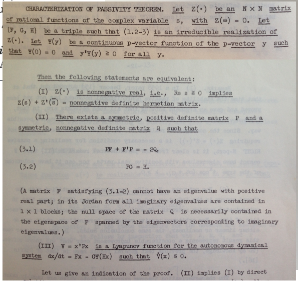
Communicated by S. Lefschetz, December 18, 1962

1. About 1950, Lur'e[1] initiated the study of a class of (closed-loop) control systems whose governing equations are

$$dx/dt = Fx - g\varphi(\sigma), \qquad d\xi/dt = -\varphi(\sigma), \qquad \sigma = h'x + \rho\xi. \qquad \text{(L)}$$

In (L), σ, ξ, ρ are real scalars, x, g, h are real
The prime denotes the transpose. F is st
real parts). $\varphi(\sigma)$ is a real-valued, continu
A_κ: $\varphi(0) = 0, 0 < \sigma\varphi(\sigma) < \sigma^2\kappa$.
We ask: Is the equ
stable) for any $\varphi \in A$
2. This problem is
(L) *is g.a.s. for every li*
crude form, however,
led to consider a more





CHARACTERIZATION OF PASSIVITY THEOREM. Let $Z(\cdot)$ be an $N \times N$ matrix of rational functions of the complex variable s, with $Z(\infty) = 0$. Let (F, G, H) be a triple such that (1.2-3) is an irreducible realization of $Z(\cdot)$. Let $\Psi(y)$ be a continuous p-vector function of the p-vector y such that $\Psi(0) = 0$ and $y'\Psi(y) \geq 0$ for all y.

Then the following statements are equivalent:

(I) $Z(\cdot)$ is nonnegative real, i.e., $\operatorname{Re} s \geq 0$ implies $Z(s) + Z'(\bar{s})$ = nonnegative definite hermetian matrix.

(II) There exists a symmetric, positive definite matrix P and a symmetric, nonnegative definite matrix Q such that

(3.1) $\qquad PF + F'P = -2Q,$

(3.2) $\qquad PG = H.$

(A matrix F satisfying (3.1-2) cannot have an eigenvalue with positive real part; in its Jordan form all imaginary eigenvalues are contained in 1×1 blocks; the null space of the matrix Q is necessarily contained in the eigenspace of F spanned by the eigenvectors corresponding to imaginary eigenvalues.)

(III) $V = x'Px$ is a Lyapunov function for the autonomous dynamical system $dx/dt = Fx - G\Psi(Hx)$ such that $\dot{V}(x) \leq 0$.

Let us give an indication of the proof. (II) implies (I) by direct

The early (miracle) years

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-16, NO. 6, DECEMBER 1971 621

Least Squares Stationary Optimal Control and the Algebraic Riccati Equation

JAN C. WILLEMS, MEMBER, IEEE

Abstract—The optimal control of linear systems with respect to quadratic performance criteria over an infinite time interval is treated. Both the case in which the terminal state is free and that in which the terminal state is constrained to be zero are treated. The integrand of the performance criterion is allowed to be fully

via the so-called Kalman–Yacubovich–Popov lemma, to the circle criterion and the Popov criterion in stability theory for feedback systems. Whereas the stability criteria just mentioned appear to be very well known, it is

Although the least squares optimization problem with linear differential constraints has roots going back to the very beginnings of calculus of variations, its revival and introduction in control theory may safely be credited to Kalman [1]. We should also mention Newton *et al.* [2], who put forward least squares techniques as a systematic basis for the design of stationary feedback control systems. Many of the results of this paper are inspired by some of the results obtained by Brockett. His work has appeared in various places in the literature and may be found in summarized form in the recent text [3]. Other papers which make ample contact with the results presented here are some of the later papers by Kalman [4], [5], the important paper by Popov [6], and the work of Anderson (see, for example, Anderson [7] and Anderson and Moore [8]).

The analytical treatment of the class of optimization problems introduced in the preceding leads to a series of matrix relations and frequency domain inequalities. Those that are important for our purposes are listed as follows. We will be interested in the case $K = K'$.

1) *The Linear Matrix Inequality (LMI):*

$$F(K) = \begin{bmatrix} A'K + KA + Q & KB + C' \\ B'K + C & R \end{bmatrix} \geq 0.$$

2) *The Quadratic Matrix Inequality (QMI):*

$$A'K + KA - (KB + C')R^{-1}(B'K + C) + Q \geq 0.$$

3) *The Algebraic Riccati Equation (ARE):*

$$A'K + KA - (KB + C')R^{-1}(B'K + C) + Q = 0.$$

4) *The Frequency-Domain Inequality (FDI):*

$$H(\bar{s}, s) = R + C(Is - A)^{-1}B + B'(I\bar{s} - A')^{-1}C' + B'(I\bar{s} - A')^{-1}Q(Is - A)^{-1}B \geq 0.$$

It is very well known that the ARE plays a crucial role in the solution of the optimal control problem under consideration. (One often gets the impression that this equation in fact constitutes the bottleneck of all of linear system theory.) However, it is much less appreciated how the other relations enter into the theory. We hope that their role will be clarified in this paper.

The seventies: realization with positivity

1972 report

a. Problems whose solution is related to positive systems.

The standard approach today to optimization, statistical filtering, and related problems utilizes the theory of the Riccati equation which was developed in system theory (primarily by the Principal Investigator) in the early 1960's. The main purpose of the present research is to provide an algebraic alternative which is in many cases more powerful and efficient than the Riccati-equation technique.

Assumptions: For simplicity, all questions discussed here will be formulated in the framework of discrete-time systems, operating over the field of real numbers. Both assumptions are inessential and are made for ease of exposition

Exact and complete behavioral data uniquely determines a minimal system whose behavior is identical with that of the given data.

$$\begin{Bmatrix}\text{system plus}\\ \text{stochastic}\\ \text{environment}\end{Bmatrix} \xrightarrow[\text{behavior}]{\text{onto}} \begin{Bmatrix}\text{behavioral}\\ \text{data}\end{Bmatrix} \xrightarrow[\text{realization}]{1-1} \begin{Bmatrix}\text{model}\end{Bmatrix}$$

(A) (B) (C)

1977 report

d. Positive Linear Systems

This area is concerned with a class of linear systems, long familiar from network synthesis, where they are characterized by "positive-real" transfer functions. The main applications envisaged were system-theoretic questions concerning the modeling of random signal sources (linear systems excited by white noise). Unfortunately, this has turned out to be the outstanding unsuccessful area of work under the Grant. The difficulty seems to be related to the fact that very little is known about the interaction between the notions "positive" and "algebraic". (Positivity: > is not an algebraic notion; inequality ≠ is.)

The main results of the earlier part of this work are summarized in the dissertation of FAURRE [1972], a former doctoral student of the Principal Investigator. Within the somewhat restricted context of FAURRE's work, the results are well known and have been extensively used in the area of stochastic modeling.

No further work on this topic is planned in the near future. The problem must be considered as unsolved at present, from the point of view of algebraic system theory.

Problem:

- characterize minimal system realizations that are also passive

Partial realization with positivity

Realization of covariance sequences:
suppose $c_0, c_1, \ldots, c_n$ are the autocorrelation lags of a stationary (zero-mean) discrete time stochastic process, i.e.,

$$c_\ell = \mathbb{E}\{y(k)y(k+\ell)\}.$$

Characterize possible extensions $c_{n+1}, c_{n+2}, \ldots$ such that

i) the series

$$F(z) := \frac{1}{2}c_0 + c_1 z^1 + c_2 z^2 + \ldots + c_n z^n + c_{n+1} z^{n+1} + \ldots$$

converges in $|z| < 1$ and positive real part (positive-real function)

ii) $F(z)$ defines a rational function of minimal degree.

Partial realization with positivity

Realization of circuits from transfer function data:
suppose $w_0, w_1, \ldots, w_n$ denote the values that the transfer function $F(z)$ takes at points $z_0, z_1, \ldots, z_n$ in $\{z \mid |z| < 1\}$.
Characterize all rational functions $F(z)$, analytic with positive real part in $|z| < 1$, such that
i)

$$F(z_i) := w_i \text{ for all } i \in \{0, 1, \ldots, n\}.$$

ii) $F(z)$ is rational function and of minimal degree.

Partial realization with positivity

Necessary & sufficient condition (existence of solutions):

$$\mathrm{Re} F(S) \geq 0$$

where S is the "compressed shift", i.e., multiplication by z^1 restricted on $H_2 \ominus B(z)H_2$

$B(z) = z^n$, or $\Pi_{i=0}^{n} \frac{z-z_i}{1-z\bar{z}_i}$

translates to:
a Toeplitz (Pick) matrix being positive semi-definite (respectively)

Will be assumed throughout

Partial realization with positivity

– **Analytic interpolation with a degree constraint**

Kalman argues that the complexity of the answer from an engineering standpoint may not be the same as that sought by physicists, information theorists, etc.

– **Applications**:

stochastic identification (Faurre, . . .)
inverse problems (recently, Baratchart, . . .)
circuit synthesis

Partial realization with positivity

REALIZATION OF COVARIANCE SEQUENCES*

R. E. Kalman

This paper examines the problem of "positivity" in relation to the partial realization of scalar power series. An exact criterion of positivity is proved for second-order realizations. The general case is currently unsolved. Even the special results contained here show that the so-called "maximum entropy principle" cannot be applied to the realization problem in the naive sense in which it is employed by physicists. It would be better to call this principle a "prejudice" because it does not fully utilize the information inherent in the data and does not provide a realization with natural ("minimal") mathematical properties.

The research I planned to report on is unfortunately not yet completed. So the following is only an outline of the problem. It has been around for a long time without receiving a definitive solution. It occupies quite a central position in system theory and has been frequently misinterpreted. It is not in the least controversial but it is unsolved.

As perhaps the only algebraist at this meeting, it is safer for me if I use a reasonably nontechnical language. In such terms, the topic of my paper is: What is the relation between partial realizations and positivity?

Evidently I must now define "partial realization" and (in relation to it) "positivity".

To take the simplest (scalar) case, consider an arbitrary sequence of numbers from a fixed field k,

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* Presented at Toeplitz Memorial Conference at Tel Aviv University, Tel Aviv, ISRAEL, on May 12, 1981. This research was supported in part under US Air Force Grant AFOSR 76-3034(D) and US Army Research Grant DAA 29-77-G-0225 through the Center for Mathematical System Theory, University of Florida.

But (20) is not sufficient to insure that the infinite sequence generated by (19) is positive! The necessary and sufficient condition for the latter requirement is (after rather extensive calculations) found to be

$$(21) \qquad |r_3| < \frac{|r_2|}{1+|r_2|} < \frac{1}{2}.$$

This is much stronger than $|r_3| < 1$. Thus, the positivity conditions on a partial sequence are not sufficient to guarantee that the corresponding minimal partial realizations generate a positive (infinite) sequence.

If it is true that the maximum entropy principle fails to reliably select the simplest (here "minimal") realization, then this alleged organizing principle of nature must contain some extraneous assumptions (which I propose to designate by the technical term prejudice, as in KALMAN [1982]). What are these extraneous assumptions? Where does the prejudice come in?

JAYNES [1968] describes the principle as follows:

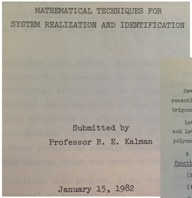
"The [prior probability assignment] that describes the available information but is maximally noncommittal with regard to the unavailable information is the one with maximum entropy (my italics)."

It is hard to quarrel with this statement on intuitive grounds. Everything hinges on the meaning given to the word "information". JAYNES and his followers apparently blindly accept that SHANNON entropy = information. But entropy is never a measure of "available information" of the mathematical type.

- how does minimality square with uniqueness?

Exact and complete behavioral data uniquely determines a minimal system whose behavior is identical with that of the given data.

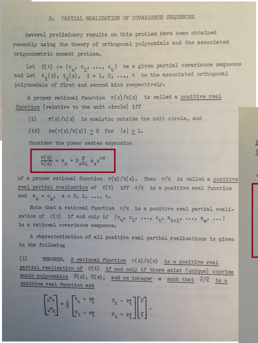
The eighties - pressing on



MATHEMATICAL TECHNIQUES FOR
SYSTEM REALIZATION AND IDENTIFICATION

Submitted by
Professor R. E. Kalman

January 15, 1982



D. PARTIAL REALIZATION OF COVARIANCE SEQUENCES

Several preliminary results on this problem have been obtained recently using the theory of orthogonal polynomials and the associated trigonometric moment problem.

Let $C(t) := \{c_0, c_1, \ldots, c_t\}$ be a given partial covariance sequence and let $\Phi_i(z)$, $\Psi_i(z)$, $i = 1, 2, \ldots, t$ be the associated orthogonal polynomials of first and second kind respectively.

A proper rational function $\pi(z)/\chi(z)$ is called a positive real function (relative to the unit circle) iff

(i) $\pi(z)/\chi(z)$ is analytic outside the unit circle, and

(ii) $\mathrm{Re}(\pi(z)/\chi(z)) \geq 0$ for $|z| \geq 1$.

Consider the power series expansion

$$\boxed{\frac{\pi(z)}{\chi(z)} = a_0 + 2\sum_{s=1}^{\infty} a_s z^{-s}}$$

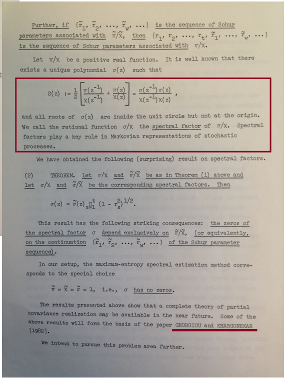
of a proper rational function $\pi(z)/\chi(z)$. Then π/χ is called a positive real partial realization of $C(t)$ iff π/χ is a positive real function and $a_s = c_s$, $s = 0, 1, \ldots, t$.

Note that a rational function π/χ is a positive real partial realization of $C(t)$ if and only if $\{c_0, c_1, \ldots, c_t, a_{t+1}, \ldots, a_w, \ldots\}$ is a rational covariance sequence.

A characterization of all positive real partial realizations is given in the following

(1) THEOREM. A rational function $\pi(z)/\chi(z)$ is a positive real partial realization of $C(t)$ if and only if there exist (unique) coprime monic polynomials $\tilde{\pi}(z)$, $\tilde{\chi}(z)$, and an integer w such that $\tilde{\pi}/\tilde{\chi}$ is a positive real function and

$$\begin{bmatrix} z^w \pi \\ z^w \chi \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \Psi_t + \Psi_t^* & \Psi_t - \Psi_t^* \\ \Phi_t - \Phi_t^* & \Phi_t + \Phi_t^* \end{bmatrix}\begin{bmatrix} \tilde{\pi} \\ \tilde{\chi} \end{bmatrix}.$$



Further, if $\{\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_w, \ldots\}$ is the sequence of Schur parameters associated with $\tilde{\pi}/\tilde{\chi}$, then $\{r_1, r_2, \ldots, r_t, \tilde{r}_1, \ldots, \tilde{r}_w, \ldots\}$ is the sequence of Schur parameters associated with π/χ.

Let π/χ be a positive real function. It is well known that there exists a unique polynomial $\sigma(z)$ such that

$$S(z) := \frac{1}{2}\left[\frac{\pi(z^{-1})}{\chi(z^{-1})} + \frac{\pi(z)}{\chi(z)}\right] = \frac{\sigma(z^{-1})\sigma(z)}{\chi(z^{-1})\chi(z)},$$

and all roots of $\sigma(z)$ are inside the unit circle but not at the origin. We call the rational function σ/χ the spectral factor of π/χ. Spectral factors play a key role in Markovian representations of stochastic processes.

We have obtained the following (surprising) result on spectral factors.

(2) THEOREM. Let π/χ and $\tilde{\pi}/\tilde{\chi}$ be as in Theorem (1) above and let σ/χ and $\tilde{\sigma}/\tilde{\chi}$ be the corresponding spectral factors. Then

$$\sigma(z) = \tilde{\sigma}(z)\prod_{s=1}^{t}(1 - r_s^2)^{1/2}.$$

This result has the following striking consequences: the zeros of the spectral factor σ depend exclusively on $\tilde{\pi}/\tilde{\chi}$, (or equivalently, on the continuation $\{\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_w, \ldots\}$ of the Schur parameter sequence).

In our setup, the maximum-entropy spectral estimation method corresponds to the special choice

$$\tilde{\pi} = \tilde{\chi} = \tilde{\sigma} = 1, \quad \text{i.e.,} \quad \sigma \text{ has no zeros}.$$

The results presented above show that a complete theory of partial covariance realization may be available in the near future. Some of the above results will form the basis of the paper GEORGIOU and KHARGONEKAR [1982].

We intend to pursue this problem area further.

Caratheodory & Nevanlinna-Pick Interpolation

– **Early insights/classical**

Parametrization of all solutions via an LFT
(Linear Fractional Transformation) $\boldsymbol{Q}$: positive real

$$\boldsymbol{F(z) = \mathrm{LFT}_{\mathrm{data}}(Q)} \qquad \text{i.e.,} = \frac{\boldsymbol{A + BQ}}{\boldsymbol{C + DQ}}$$

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$$\mathrm{Re}\boldsymbol{F}(e^{j\theta}) \sim \mathrm{Re}\boldsymbol{Q}(e^{j\theta})$$

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– **Classical/early insights**

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$$\mathrm{Re}\boldsymbol{F(e^{j\theta})} \sim \mathrm{Re}\boldsymbol{Q(e^{j\theta})}$$

also,

$$\boldsymbol{F} \text{ rational } \Leftrightarrow \boldsymbol{Q} \text{ rational}$$

Georgiou-Khargonekar 1981

Caratheodory & Nevanlinna-Pick Interpolation

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Parametrization of all solutions via an LFT
(Linear Fractional Transformation) $\boldsymbol{Q}$: positive real

$$\boldsymbol{F}(z) = \mathbf{LFT}_{\mathbf{data}}(\boldsymbol{Q}) \qquad \text{i.e.,} = \frac{\boldsymbol{A} + \boldsymbol{B}\boldsymbol{Q}}{\boldsymbol{C} + \boldsymbol{D}\boldsymbol{Q}}$$

However,

$$|\boldsymbol{n} - \mathbf{deg}\boldsymbol{Q}| \leq \mathbf{deg}\boldsymbol{F} \leq \boldsymbol{n} + \mathbf{deg}\boldsymbol{Q}$$

How to choose $\boldsymbol{Q}$?
Algebraic characterization unkown?

Analytic Interpolation with degree constraint

– **Topological constraints (Georgiou 1983, 1987)**

Degree n solutions via Homotopy methods

For any polynomial η of degree $\leq n$,
there exists F of degree n, i.e., $F = \pi/\chi$,
π, χ polynomials of degree $\leq n$, such that

$$\mathrm{Re} F(z) = \frac{|\eta(z)|^2}{|\chi(z)|^2} \text{ for } z = e^{i\theta}$$

is positive real and satisfies $n+1$ interpolation constraints

η : "spectral numerator"
moving average part of an f.d. stochastic realization

Analytic Interpolation with degree constraint

– **Generic cases (Georgiou 1983):**

Both data sets:

$$\{\boldsymbol{c_i's} \text{ or } \boldsymbol{w_i's} \mid \text{ minimal realization has degree } = \boldsymbol{n}\}$$

as well as the opposite,

$$\{\boldsymbol{c_i's} \text{ or } \boldsymbol{w_i's} \mid \text{ minimal realization has degree } < \boldsymbol{n}\}$$

have open interiors (generic).

$\Rightarrow$ **Kalman's problem has a non-trivial set of solutions**

1990's & 2000's - state of the problem

Via optimization & other methods:

Complete parametrization

$$\{\text{solutions of degree } \leq n\} \leftrightarrow \{\eta \mid \text{ stable of degree } \leq n\}$$

Byrnes-Lindquist-Gusev-Matveev 1995 (uniqueness)
Byrnes-Landau-Lindquist 1997
Burnes-Gusev-Lindquist 1998 (KL-optimization)
Georgiou 1999
Byrnes-Georgiou-Lindquist 2001, 2003,...
Byrnes-Georgiou-Lindquist-Megretski 2006
Takyar-Georgiou 2007 (matrix-valued)

1990's & 2000's - state of the problem

Byrnes-Georgiou-Lindquist-Megretski (Trans. AMS, 2004)
Generalized interpolation in H^∞ with a complexity constraint

$K := H_2 \ominus B(z)H_2$ with B arbitrary inner (i.e., B is "all-pass")
T bounded operator on K, commutes with S

$\mathrm{Re}(T) \geq 0$ (necessary condition)

$\exists$ solutions $F = \pi/\chi$ with $\pi, \chi \in K$, i.e., $F(S) = T$

$$\text{any such solution} \leftrightarrow \eta \in K \text{ via } \mathrm{Re}F = \frac{|\eta|^2}{|\chi|^2}$$

Further F is the unique maximizer of

$$F \mapsto \int |\eta|^2 \log(\mathrm{Re}F)$$

state of the problem

– Complete parametrization of degree $\leq n$ solutions

Smooth parametrization $\leftrightarrow$ stable spectral numerator
Multivariable interpolation (F matrix-valued)
Applications (spectral analysis, inverse problems)

Kalman's problem ($< n$?) remains open:
algebraic characterization of minimal degree
positive-real interpolants

when is the minimal degree $< n$?
what is the minimal degree?
are there suitable algebraic invariants?

A tribute to Professor Rudolf E. Kalman

Gainesville, June 12, 2016