# MAE Preliminary Examination

**Mathematics Section**

**Monday, November 13, 2023, 9:00am-11:30noon**

| **Your Name** |
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| |

| THREE PROBLEMS WILL BE GRADED<br>**Select the 3 problems you've worked, to be graded:** | **points** |
| --- | --- |
| **Problem:** | /10 |
| **Problem:** | /10 |
| **Problem:** | /10 |
| **Total** | /30 |

**Please give your answers/work in the space provided**

Explain your work/steps clearly

**Linear Algebra: Problem 1 [10 points]:**

1a) Consider a system of linear time-invariant ordinary differential equations

$$\dot{x}(t) = Sx(t), \text{ where } x(t) \in \mathbb{R}^n$$

for $n$ being a positive integer, and $S$ a skew-symmetric $n \times n$ matrix; that is, $S + S^T = 0$, where $^T$ denotes transpose. Show that the trajectory of the system that starts from an initial condition $x(0)$ on the unit sphere, i.e., the Euclidean norm being $\|x(0)\| = 1$, remains on the unit sphere for all times. In other words, you need to prove that $\|x(t)\| = 1$ for all $t$.

1b) Consider a square matrix $Q \in \mathbb{R}^{n\times n}$ such that for any $x \in \mathbb{R}^n$, the Euclidian norm of $Qx$ is the same as that of $x$, i.e., $\|Qx\| = \|x\|$. What can you deduce about the matrix $Q$? Specifically, you need to specify what is the value of the determinant of $Q$, what is the inverse of $Q$, and where are the eigenvalues of $Q$ located on the complex plane.

Workspace for Problem 1: Explain your reasoning/work here.

**Solution:**

**1a)** One can argue in several different (equivalent) ways:

**Solution i)** The solution of the ODE is $x(t) = e^{St}x(0)$, and one can readily see that

$$\begin{aligned}(e^{St})^T e^{St} &= e^{S^T t} e^{St} \\ &= e^{-St} e^{St} = e^{(-S+S)t} \\ &= e^{\text{zero matrix}} = \text{identity matrix}.\end{aligned}$$

Therefore $e^{St}$ is an orthogonal matrix for any $t$. It follows that

$$\begin{aligned}\|x(t)\|^2 &= (e^{St}x(0))^T e^{St}x(0) \\ &= x(0)^T e^{S^T t} e^{St} x(0) \\ &= x(0)^T x(0) = 1.\end{aligned}$$

**Solution ii)** We consider the derivative of $\|x(t)\|^2$ and show that this is equal to zero, and therefore, that the length of $x(t)$ remains constant. To this end, we compute

$$\begin{aligned}\frac{d}{dt} x(t)^T x(t) &= \dot{x}(t)^T x(t) + x(t)^T \dot{x}(t) \\ &= x(t)^T S^T x(t) + x(t)^T S x(t) \\ &= x(t)^T (S^T + S) x(t) = 0.\end{aligned}$$

**1b)** Since $\|Qx\| = \|x\|$ for all $x$,

$$x^T Q^T Q x = x^T x,$$

and therefore $Q^T Q = I$, the identity matrix. Therefore, $Q$ is an orthogonal matrix. It can be easily seen that

$$\det(Q^T Q) = (\det Q^T)(\det Q) = 1,$$

and therefore $\det Q = 1$. Also, $Q^{-1} = Q^T$, and lastly, since

$$Qx = \lambda x$$

and $x$ have the same Euclidean length, for any eigenvalue/eigenvector pair, i.e.,

$$\|x\|^2 = (Qx)^* Qx = \overline{(Qx)^T} Qx = |\lambda|^2 \|x\|^2,$$

the eigenvalues must have $|\lambda|^2 = 1$, i.e., they lie on the unit circle of the complex plane.

Workspace for Problem 1: Explain your reasoning/work here.

**Solution:**

## Differential Equations: Problem 2 [10 points]:

Consider the second-order differential equation

$$\ddot{x}(t) + \dot{x}(t) - x(t) + x(t)^2 = 0.$$

i) Write a corresponding state-space representation in the form of two first order differential equations, with position $x(t)$ and velocity $\dot{x}(t)$ as state variables, as you will consider the dynamics on the phase plane. Having done that, determine:

ii) critical points, i.e., points of equilibrium.

iii) linearization of the differential equations about each point of equilibrium

iv) the eigenvalues and any real eigenvectors of the corresponding linearized models

v) the type of critical point each is (e.g., stable, unstable, focus, saddle point).

vi) draw as best as you can the phase portrait. It is important to indicate correctly the rotation about foci, i.e., clockwise or counterclockwise.

**Solution:**

i) with

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

we get the state-space representation

$$\begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_2 \\ -\mathbf{x}_2 + \mathbf{x}_1 - \mathbf{x}_1^2 \end{pmatrix} =: f(\mathbf{x}).$$

ii) For $f(\mathbf{x}) = 0$ we get that $\boxed{\mathbf{x}_2 = \dot{x} = 0 \text{ while } \mathbf{x}_1 \in \{0, 1\}}$, for two points of equilibrium, respectively.

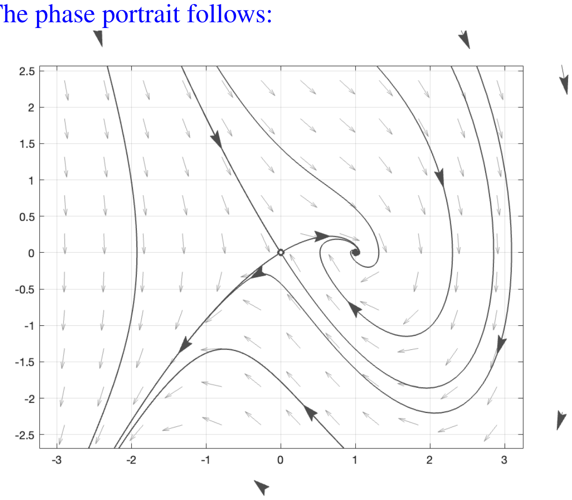
iii-v) The Jacobian is

$$\partial f / \partial \mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 - \mathbf{2x}_1 & -1 \end{pmatrix}.$$

Thus, at:

| equilibrium point | linearized dynamics | eigenvalues/vectors | type of equilibrium |
|---|---|---|---|
| $\mathbf{x}_{\text{eq. 1}} = \binom{0}{0}$: | $\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x}$ | $\lambda_{1,2} = \frac{-1 \mp \sqrt{5}}{2}$, $v_{1,2} \in \{\binom{-0.52}{0.85}, \binom{0.85}{0.52}\}$ | saddle |
| $\mathbf{x}_{\text{eq. 2}} = \binom{1}{0}$: | $\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ \mathbf{-1} & -1 \end{pmatrix} \mathbf{x}$ | $\lambda_{1,2} = \frac{-1 \pm i\sqrt{\mathbf{3}}}{2}$, $v_{1,2}$ complex | stable focus |

v) The phase portrait follows:




To verify clockwise rotation about the stable focus, observe that to the right of the particular point of equilibrium, i.e., for $\dot{x} = 0$ and $x > 1$, it holds that $\ddot{x} < 0$, i.e., $\dot{\mathbf{x}}_2 < 0$ and the vector field points towards decreasing values of $\mathbf{x}_2$.

Workspace for Problem 2: Explain your reasoning/work here.

**Solution:**

**Problem 3 [10 points]:**

Consider the wave equation

$$u_{tt} - a^2 \Delta u = 0$$

in the cubic domain $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$, where $a$ is a positive constant.

The boundary conditions are

$$\frac{\partial u}{\partial n} = 0 \text{ on five sides of the cube at } x = 0,\, x = 1,\, y = 0,\, y = 1, \text{ and } z = 0$$

and

$$u = 0 \text{ on } z = 1$$

The initial conditions are

$$u(0, x, y, z) = (1 - 2y)^2 cos(\pi z/2)$$

and

$$u_t(0, x, y, z) = (1 - 2z)^2$$

Using separation of variables and eigenfunction expansion, find $u(t, x, y, z)$. The coefficients of series expansions can be left in integral form. You may notice that the initial and boundary conditions are uniform in the $x$ direction.

Workspace for Problem 3: Explain your reasoning/work here.

**Solution:**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

Assume

$$u = \sum \sum \sum u_{lmn}$$

$$u_{lmn} = X(x)Y(y)Z(z)T(t)$$

For linear equation

$$\frac{\partial^2 u_{lmn}}{\partial t^2} = a^2 \nabla^2 u_{lmn}$$

Differentiate and divide by $u_{lmn}$

$$\frac{T''}{T} = a^2 \left( \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} \right)$$

$$Z_n'' + \nu_n^2 Z_n = 0; \quad Z_n'(0) = 0, \quad Z_n(1) = 0$$

$$Z_n = \cos \nu_n z$$

$$\nu_n = (2n + 1)\frac{\pi}{2}, \quad n = 0, 1, 2, \cdots$$

$$Y_m'' + \mu_m^2 Y_m = 0; \quad Y_m'(0) = 0, \quad Y_m'(1) = 0$$

$$Y_m = \cos \mu_m z$$

$$\mu_m = m\pi, \quad m = 0, 1, 2, \cdots$$

$$X_l'' + \lambda_l^2 X_l = 0; \quad X_l'(0) = 0, \quad X_l'(1) = 0$$

$$X_l = \cos \lambda_l z$$

$$\lambda_l = l\pi, \quad l = 0, 1, 2, \cdots$$

$$T'' + a^2(\lambda_l^2 + \mu_m^2 + \nu_n^2)T = 0$$

$$T = \sin \omega_{lmn} t, \cos \omega_{lmn} t$$

where

$$\omega_{lmn} = a\sqrt{\lambda_l^2 + \mu_m^2 + \nu_n^2} = a\pi\sqrt{l^2 + m^2 + [(2n+1)/2]^2}$$

By the principle of superposition, we have

$$u = \sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}(A_{lmn}\cos\omega_{lmn}t + B_{lmn}\sin\omega_{lmn}t)\cos\lambda_l x\cos\mu_m y\cos\nu_n z$$

Using the initial conditions, we get

$$u(0,x,y,z) = \sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}A_{lmn}\cos\lambda_l x\cos\mu_m y\cos\nu_n z = (1-2y)^2 cos(\pi z/2)$$

$$\frac{\partial u}{\partial t}(0,x,y,z) = \sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\omega_{lmn}B_{lmn}\cos\lambda_l x\cos\mu_m y\cos\nu_n z = (1-2z)^2$$

$A_{lmn} = 0$ for all $l$ and $n$ except $l = 0$ and $n = 0$.
$B_{lmn} = 0$ for all $l$ and $m$ except $l = 0$ and $m = 0$.
Denoting

$$A_m = A_{0m0}$$

$$B_n = B_{00n}$$

$$\omega_m = \omega_{0m0} = a\pi\sqrt{m^2 + (1/2)^2}$$

and

$$\omega_n = \omega_{00n} = a\pi\frac{2n+1}{2}$$

we have

$$\begin{aligned} u(t,x,y,z) &= \cos(\pi z/2)\sum_{m=0}^{+\infty}A_m\cos\omega_m t\cos\mu_m y + \sum_{n=0}^{+\infty}B_n\sin\omega_n t\cos\nu_n z \\ &= \cos(\pi z/2)\sum_{m=0}^{+\infty}A_m\cos(a\pi\sqrt{m^2+(1/2)^2}t)\cos(m\pi y) + \sum_{n=0}^{+\infty}B_n\sin[a(2n+1)\frac{\pi}{2}t]\cos[(2n+1)\frac{\pi}{2}z] \end{aligned}$$

where

$$A_0 = \int_0^1 (1-2y)^2 dy$$

$$A_m = 2\int_0^1 (1-2y)^2\cos\mu_m y dy, \quad m = 1, 2, \cdots,$$

$$B_n = \frac{2}{\omega_n}\int_0^1 (1-2z)^2\cos\nu_n z dz \quad n = 1, 2, \cdots,$$

Essentially, we have two solutions, one from the two-dimensional initial displacement condition $u(t=0) = f(y)\cos(\pi z/2)$, and the other for the one-dimensional initial velocity condition $\frac{\partial u}{\partial t}(t=0) = g(z)$

Alternatively, one may simplify the above solution procedure if we start by solving a 2-dimensional problem in the $y$-$z$ plane on recognizing the independence of the problem from the $x$ coordinate. The final solution will, of course, be the same.

Workspace for Problem 3: Explain your reasoning/work here.

**Solution:**

**Problem 4 [10 points]:**
We have the heat equation

$$u_t - \alpha u_{xx} = 0 \quad \text{in the finite domain } 0 \le x \le 1 \text{ and } t \ge 0$$

with the initial condition: $u(0,x) = 0$ and the boundary conditions: $u(t,0) = h_0$ and $u(t,1) = 0$, where $h_0$ is a constant. Solve for $u(t,x)$ using Laplace transform. Define any integral function that use.

You may find the following series and Laplace Transform Table useful.

$$\frac{1}{1-\epsilon} = \sum_{n=0}^{+\infty} \epsilon^n, \quad \text{for } |\epsilon| < 1$$

Table 1: Some Useful Laplace Transforms

| $f(t)$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t^n, \quad n = 0, 1, 2, \cdots$ | $\frac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\frac{1}{s-a} \quad s > a$ |
| $\sin at$ | $\frac{a}{s^2+a^2} \quad s > 0$ |
| $\cos at$ | $\frac{s}{s^2+a^2} \quad s > 0$ |
| $H(t-a)$ | $\frac{e^{-as}}{s} \quad s > 0$ |
| $H(t-a)f(t-a)$ | $e^{-as}F(s)$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ | $F(s)G(s)$ |
| $f^n(t)$ | $s^nF(s) - s^{n-1}f(0) - \cdots - f^{n-1}(0)$ |
| $\int_0^t f(\tau)d\tau$ | $\frac{1}{s}F(s)$ |
| $\text{erf}(t/2a)$ | $\frac{1}{s}e^{a^2s^2}\,\text{erfc}(as)$ |
| $\text{erfc}(\frac{a}{2\sqrt{t}})$ | $\frac{1}{s}e^{-a\sqrt{s}}$ |
| $\frac{1}{\sqrt{\pi t}}$ | $1/\sqrt{s}$ |
| $\frac{1}{\sqrt{\pi t}}e^{\frac{-a^2}{4t}}$ | $e^{-a/s}/\sqrt{s}$ |
| $t^{-1/2}e^{-a^2/4t}$ | $\sqrt{\frac{\pi}{s}}e^{-a\sqrt{s}} \quad a \ge 0$ |
| $t^{-3/2}e^{-a^2/4t}$ | $\frac{2\sqrt{\pi}}{a}e^{-a\sqrt{s}} \quad a > 0$ |
| $2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{a^2}{4t}\right) - a\,\text{erfc}\,\frac{a}{2\sqrt{t}}$ | $\frac{1}{s^{3/2}}e^{-a\sqrt{s}} \quad a \ge 0$ |

Workspace for Problem 4: Explain your reasoning/work here.

**Solution:**
Performing the Laplace transform on the PDE and the boundary conditions we get

$$sU - \alpha U_{xx} = 0$$

where $U(x,s) = \mathcal{L}u$ with the boundary conditions

$$U(0,s) = h_0/s$$

$$U(1,s) = 0$$

The solutions is

$$U(x,s) = \frac{h_0}{s}\frac{e^{-\sqrt{\frac{s}{\alpha}}x} - e^{-\sqrt{\frac{s}{\alpha}}(2-x)}}{1 - e^{-2\sqrt{\frac{s}{\alpha}}}}$$

$$U(x,s) = \frac{h_0}{s}\left(e^{-\frac{x}{\sqrt{\alpha}}\sqrt{s}} - e^{-\frac{2-x}{\sqrt{\alpha}}\sqrt{s}}\right)\sum_{n=0}^{+\infty} e^{-\frac{2n}{\sqrt{\alpha}}\sqrt{s}} = h_0\sum_{n=0}^{+\infty}\left(\frac{1}{s}e^{-\frac{x+2n}{\sqrt{\alpha}}\sqrt{s}} - \frac{1}{s}e^{-\frac{2(n+1)-x}{\sqrt{\alpha}}\sqrt{s}}\right)$$

Notice the inverse transform

$$\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-a\sqrt{s}}\right\} = \mathrm{erfc}(\frac{a}{2\sqrt{t}})$$

We get

$$u(x,t) = h_0 \sum_{n=0}^{+\infty}\left[\mathrm{erfc}\left(\frac{x+2n}{\sqrt{4\alpha t}}\right) - \mathrm{erfc}\left(\frac{2(n+1)-x}{\sqrt{4\alpha t}}\right)\right]$$

where $\mathrm{erfc}(z)$ is the complimentary error function

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{+\infty} e^{-\eta^2}\,d\eta$$

Thus,

$$u(x,t) = \frac{2h_0}{\sqrt{\pi}}\sum_{n=0}^{\infty}\left[\int_{\frac{x+2n}{\sqrt{4\alpha t}}}^{\infty} e^{-\eta^2}\,d\eta - \int_{\frac{2(n+1)-x}{\sqrt{4\alpha t}}}^{\infty} e^{-\eta^2}\,d\eta\right] = \frac{2h_0}{\sqrt{\pi}}\sum_{n=0}^{\infty}\int_{\frac{x+2n}{\sqrt{4\alpha t}}}^{\frac{2(n+1)-x}{\sqrt{4\alpha t}}} e^{-\eta^2}\,d\eta$$

Workspace for Problem 4: Explain your reasoning/work here.

**Solution:**