MAE Preliminary Examination

Mathematics Section

Monday, November 14, 9:00am-12noon

Your Name

THREE PROBLEMS WILL BE GRADED **Select the 3 problems you've worked, to be graded:**	**points**
Problem:	$/10$
Problem:	$/10$
Problem:	$/10$
Total	$/30$

Please give your answers/work in the space provided

Explain your work/steps clearly

Linear Algebra: Problem 1 [10 points]:

i) Consider the 2×2 matrix

$$P = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}.$$

Determine a matrix S such that $P = SS'$.

ii) Explain in general, for a symmetric matrix P when it is possible to obtain such a factorization $P = SS'$ and how this can be accomplished.

iii) Suppose that a matrix A and its transpose A' satisfy an equation of the form

$$PA = A'P$$

for a positive definite matrix P. Show that A has real simple eigenvalues. (Hint: Consider searching for a similarity transformation that makes $B = SAS^{-1}$ symmetric.)

Workspace for Problem 1: Explain your reasoning/work here.

Solution:

i) Although this can be accomplished in a brute force way, by searching of such an algebraic factorization directly since this is only 2×2, a systematic way is to compute the square root of the matrix just as we would do for any other function of the matrix (by spectral decomposition). Specifically we first compute the eigendecomposition

$$P = \begin{bmatrix} -1.618 & 1 \\ 1 & 1.618 \end{bmatrix} \begin{bmatrix} \frac{7-\sqrt{45}}{2} & 0 \\ 0 & \frac{7+\sqrt{45}}{2} \end{bmatrix} \begin{bmatrix} -1.618 & 1 \\ 1 & 1.618 \end{bmatrix}^{-1}$$

and then construct the square root of P setting

$$S = \begin{bmatrix} -1.618 & 1 \\ 1 & 1.618 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{7-\sqrt{45}}{2}} & 0 \\ 0 & \sqrt{\frac{7+\sqrt{45}}{2}} \end{bmatrix} \begin{bmatrix} -1.618 & 1 \\ 1 & 1.618 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

ii) The construction in i) is general. I.e., we start with an eigendecomposition

$$P = VDV^{-1}$$

where D is diagonal and, as long as $P > 0$, we can take the square roots of the diagonal elements (eigenvalues) in D to construct

$$S = V\sqrt{D}V^{-1}.$$

iii) It is well known and easy to show that any symmetric matrix has real eigenvalues. (To see this, simply consider $v^*Av = \lambda v^*v$ with v the eigenvector corresponding to the eigenvalue λ. Taking complex conjugates of both sides, $\lambda = \bar{\lambda}$ and hence, real.)

So we only need to show that A can be transformed to a symmetric matrix via a similarity transformation. For then, $B = SAS^{-1} = B'$ will have real eigenvalues, which would be identical to those of A.

To see this last statement in the hint, simply consider the factorization $P = SS$ as before, since $P > 0$. (We could also use $P = SS'$, with the extra complication of carrying primes in the needed computations.) Then

$$SSA = A'SS \;\Rightarrow B = SAS^{-1} = S^{-1}A'S = B'$$

which is exactly what we wanted to prove.

Differential Equations: Problem 2 [10 points]:

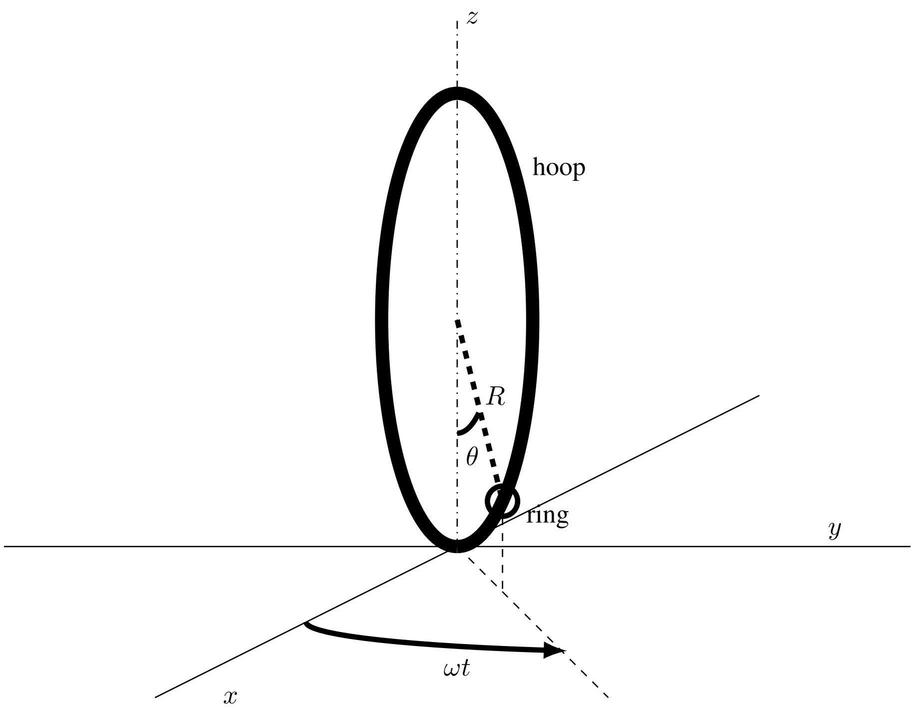

Figure 1: Rotating hoop with a sliding ring/bead.

Consider the physical system shown above that consists of a hoop (big thick circle) that rotates about the z axis with angular velocity ω. The angular position of its plane with respect to the x axis is indicated by ωt, where t denotes time. A ring/bead (small circle denoted "ring") is placed on the hoop which can freely move up and down with a small amount of friction. The angular location of this ring/bead with respect to the center of the hoop is denoted by θ. The equation of motion that dictates the position θ of the ring, as the hoop rotates, is given by

$$\ddot{\theta} + \mu\dot{\theta} + (g/R)\sin(\theta) - \omega^2 \sin(\theta)\cos(\theta) = 0,$$

where μ can be considered constant. Do the following:

i) Write down the dynamical equation in state-space form. [Hint: set $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$, and treat μ, g, R, ω as constants.]
ii) Determine the angular positions for the ring/bead that correspond to points of equilibrium. I.e., determine values of θ where the bead can be found at steady-state for a fixed angular velocity ω of the hoop. These points may depend on the value of the angular velocity ω. Clearly, one of them is $\theta = 0$ but there may be one more.
iii) Classify these possible points of equilibrium (i.e., stable, unstable, node, focus, etc.).

[Hint: the stability of these points of equilibrium may depend on the value of ω for fixed values of g, R. For instance, $\theta_0 = 0$ is one angle that corresponds to an equilbrium, but this may not be stable depending on the value of ω.]

Above, $R > 0$ is the radius of the hoop, $\mu > 0$ is a coefficient of friction, and $g = 9.81[\mathrm{m}/\mathrm{sec}^2]$ denotes the acceleration due to gravity, and note that the dynamics of the ring are second order (since ω is thought of being constant) and the dynamical equation involves only the second derivative of θ.

Workspace for Problem 2: Explain your reasoning/work here.

Solution:

i) For $x_1 = \theta$, $x_2 = \dot{\theta}$, the dynamical equation is

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{g}{R}\sin(x_1) + \omega^2 \sin(x_1)\cos(x_1) - \mu x_2 \end{pmatrix} = f(x_1, x_2).$$

ii) To determine points of equilibrium, we need $\dot{\theta} = \ddot{\theta} = 0$, i.e., $\dot{x}_1 = \dot{x}_2 = 0$,

$$(g/R)\sin(\theta) - \omega^2 \sin(\theta)\cos(\theta) = 0.$$

Hence, there are points of equilibrium at $\boxed{\theta_0 = 0 \text{ and } \theta_1 = \pi}$,

and, possibly, an additional point of equilibrium at $\boxed{\theta_2 = \arccos(\frac{g}{R\omega^2})}$, but only when $g < R\omega^2$.

iii$_0$) For $\theta_0 = 0$, i.e., $\boxed{(x_1, x_2) = (0, 0)}$ and the linearized dynamics $\dot{x} = Ax$ correspond to:

$$\boxed{A = \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{R} + \omega^2 & -\mu \end{pmatrix}}$$

Thus, if $\omega = 0$, and if $\mu^2 > 4g/R$, then the equilibrium at θ_0 is a stable node,
if $\mu^2 < 4g/R$, then it is a stable focus (damped oscillations).
The characteristic exponent

$$\frac{-\mu + \sqrt{\mu^2 - 4(g/R) + 4\omega^2}}{2}$$

is negative when $R\omega^2 < g$, i.e.,

$$\boxed{\omega < \sqrt{g/R}}.$$

If this holds, the equilibrium is stable, otherwise it is unstable (and marginally so, if $\omega = \sqrt{g/R}$).

The equilibrium is a **stable focus** as long as, in addition to stability, $\boxed{\mu < 2\sqrt{g/R - \omega^2}}$,
otherwise it is a **stable node**.
As we increase ω further, the characteristic exponent becomes zero when $\boxed{\omega = \sqrt{g/R}}$, and for $\omega > \sqrt{g/R}$, the equilibrium at $\theta_0 = 0$ is an unstable focus. For such values of ω, θ_2 becomes an attractive equilibrium (below).

iii$_1$) At $\theta_1 = \pi$, the equilibrium is clearly unstable. The state matrix at the corresponding equillibrium is

$$\boxed{A = \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ \frac{g}{R} + \omega^2 & -\mu \end{pmatrix}}$$

iii$_2$) For $\theta_2 = \arccos(\frac{g}{R\omega^2})$, i.e., $\boxed{(x_1, x_2) = (\arccos(\frac{g}{R\omega^2}), 0)}$ the linearized dynamics correspond to

$$A = \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ \frac{g^2}{R^2\omega^2} - \omega^2 & -\mu \end{pmatrix}$$

$\boxed{\text{The equilibrium (exists and it) is stable for } \omega > \sqrt{g/R} \text{ and } \mu > 0.}$

$\boxed{\text{The equilibrium is a } \textbf{node} \text{ for } \mu^2 + 4(\frac{g^2}{R^2\omega^2} - \omega^2) > 0, \text{ and a } \textbf{focus} \text{ otherwise.}}$

Problem 3 [10 points]:

Solve the PDE

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

in the domain $0 \le r \le 1$, $0 \le \theta \le \pi$, and $t \ge 0$ with the following boundary and initial conditions

$$u(0, r, \theta) = f(r) \quad (1)$$

$$\frac{\partial u}{\partial t}(0, r, \theta) = 0 \quad (2)$$

$$\frac{\partial u}{\partial r}(t, 1, \theta) = 0 \quad (3)$$

$$\frac{\partial u}{\partial \theta}(t, r, 0) = 0 \quad (4)$$

$$\frac{\partial u}{\partial \theta}(t, r, \pi) = 0 \quad (5)$$

where $f(r)$ is given. Identify the eigenvalue/eigenfunction problem. Determine the eigenfunctions, eigenvalues, and the expansion of the solution in terms of the
eigenfunction series. Express the coefficients in the series of integrals of $f(r)$. Include the case where $\int_0^1 r f(r) dr \neq 0$.

Workspace for Problem 3: Explain your reasoning/work here.

Solution:

In polar coordinate system, the PDE becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right]$$

The wave speed c is kept here for generality.

We use separation of variables, following pretty much the same approach as for Problem 2 in Homework #5.

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t)$$

Recognizing that this problem has no dependence on θ either in the boundary conditions or the initial conditions, we infer that only the constant function $\Theta_n(\theta) = 1$ is needed or we can eliminate the θ dependence in the above PDE completely and considered the one dimensional problem below.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right]$$

We then assume

$$u(r, \theta, t) = R(r)T(t)$$

Plugging this into our equation yields,

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2$$

where λ must be a constant. The problem is then converted to 2 ODE problems.

$$\ddot{T} + c^2\lambda^2 T = 0$$

$$r^2 R'' + rR' + r^2\lambda^2 R = 0$$

$$|R(0)| < +\infty, \quad R'(1) = 0$$

The second problem for $R(r)$ is a Sturm-Liouville eigenvalue problem. Application of the Rayleigh quotient shows the eigenvalue $\lambda^2 \ge 0$ (in fact $\lambda > 0$ for $n > 0$ as we will see below). Therefore, we do not need to check the case when $\lambda^2 < 0$. But we do need to check on the case when λ might be zero.

Case 1) $\lambda = 0$

$$r^2 R'' + rR' = 0$$

$$R(r) = A + B \ln r$$

Application of the zero Neumann BCs yields $B = 0$ and the constant function $R(r) = 1$ as one eigenfunction.

Case 2) For $\lambda > 0$, we know the solution to the ODE is

$$R(r) = AJ_0(r\lambda) + BY_0(r\lambda)$$

Since $R(r)$ must be bounded. This determines that $B = 0$. Now applying the BC on the edge of the disk gives

$$R'(1) = AJ_0'(\lambda) = 0$$

which is possible when λ is a root for $J_0'(x) = 0$. We know $J_0(x)$ is oscillatory with an infinite number of zeros and in between every pair of consecutive zeros there is a root for $J_0'(x) = 0$. See Figure 2.

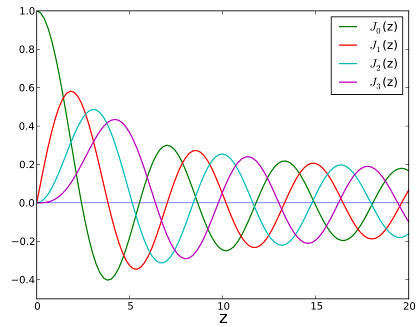

Figure 2: Bessel function $J_n(x)$.

Summarizing both the $\lambda = 0$ and $\lambda > 0$ cases, we get the following full set of eigenvalue and eigenfunction pairs.

$$\begin{array}{ll} \lambda = 0, & R_0(r) = 1 \\ \lambda = \lambda_m = z'_{0,m}, & R_m(r) = J_0(z'_{0,m}r), \quad m = 1, 2, ..., \end{array}$$

where $z'_{0,m}$ is the m-th root of $J_0'(z)$ (see Table 2 in my Notes on Bessel functions). By Sturm-Liouville theorem, we know these eigenfunctions are orthogonal for the inner product definition $< f, g >= \int_0^1 f(r)g(r)\sigma(r)dr$ with the weight function $\sigma(r) = r$.

Now, we can get the corresponding time equation solution for each of the eigenfunctions found above.

$$\ddot{T} + \nu\lambda^2 T = 0$$

$$T(t) = \begin{cases} A_0 + B_0 t & (\lambda = 0) \\ A_m \cos(c\lambda_m t) + B_m \sin(c\lambda_m t) & (\lambda_m = z'_{0,m} > 0, m = 1, 2, \ldots) \end{cases}$$

where A_m, B_m are constants. Our final solution can be expressed in terms of the following series.

$$u(r, \theta, t) = A_0 + B_0 t + \sum_{m=1}^{\infty} [A_m \cos(cz'_{0,m}t) + B_m \sin(cz'_{0,m}t)] J_0(z'_{0,m}r)$$

Since the initial velocity $u_t = 0$, we immediately get $B_m = 0$ for all $m = 0, 1, ldots$. Application for the initial displacement condition gives

$$A_0 = \frac{\int_0^1 f(r) r dr}{\int_0^1 r dr} = 2\int_0^1 f(r) r dr, \quad A_m = \frac{\int_0^1 f(r) J_0(z'_{0,m}r) r dr}{\int_0^1 J_0^2(z'_{0,m}r) r dr}$$

where $f(r) = (1 - r^2)$, given in the problem. The final solution is

$$u(r, \theta, t) = A_0 + \sum_{m=1}^{\infty} A_m J_0(z'_{0,m}r) \cos(\omega_m t)$$

Problem 4 [10 points]:

Use the Fourier transform to solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

in the infinite domain $-\infty < x < \infty$ for $t > 0$ given that $u(0, x) = f(x)$, where

$$\begin{aligned} f(x) &= 0 \quad \text{for} \quad -\infty < x < -1 \\ f(x) &= 1 \quad \text{for} \quad -1 \leq x < 0 \\ f(x) &= -1 \quad \text{for} \quad 0 \leq x < 1 \\ f(x) &= 0 \quad \text{for} \quad 1 \leq x < \infty \end{aligned}$$

Workspace for Problem 3: Explain your reasoning/work here.

Solution:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ u(x,0) = f(x) = H(x+1) - 2H(x) + H(x-1) \end{cases}$$

Let $U(\omega, t) = \mathcal{F}\{u(x,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t) e^{i\omega x} dx$

$F(\omega) = \mathcal{F}(f(x))$

Apply $\mathcal{F}$ to the PDE and Initial Conditions: $\begin{cases} \frac{d}{dt} U = -\omega^2 U \\ U(\omega, 0) = F(\omega) \end{cases}$

Solution: $U = F(\omega) e^{-\omega^2 t}$

$$u(x,t) = \mathcal{F}^{-1}\{F(\omega) e^{-\omega^2 t}\} = f(x) * \mathcal{F}^{-1}\{e^{-\omega^2 t}\}$$

$$= f(x) * \frac{1}{\sqrt{4\pi t}} e^{-\omega^2/4t}$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(x-\xi) e^{-\xi^2/4t} d\xi$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x - \sqrt{4t}\,\eta) e^{-\eta^2} d\eta, \quad \text{where } \eta = \xi/\sqrt{4t}$$

Notice $\int_{-\infty}^{\infty} H(x - x_0 - \sqrt{4t}\,\eta) e^{-\eta^2} d\eta = \int_{-\infty}^{\frac{x-x_0}{\sqrt{4t}}} e^{-\eta^2} d\eta$

$$= \int_{-\infty}^{0} e^{-\eta^2} d\eta + \int_{0}^{\frac{x-x_0}{\sqrt{4t}}} e^{-\eta^2} d\eta$$

$$= \frac{\sqrt{\pi}}{2}\left[1 + \frac{2}{\sqrt{\pi}} \int_0^{(x-x_0)/\sqrt{4t}} e^{-\eta^2} d\eta\right] = \frac{\sqrt{\pi}}{2}\left[1 + \operatorname{erf}\left(\frac{x-x_0}{\sqrt{4t}}\right)\right]$$

$$u(x,t) = \frac{1}{2}\left\{\left[1 + \operatorname{erf}\left(\frac{x+1}{\sqrt{4t}}\right)\right] - 2\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{4t}}\right)\right] + \left[1 + \operatorname{erf}\left(\frac{x-1}{\sqrt{4t}}\right)\right]\right\}$$

$$= \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+1}{\sqrt{4t}}\right) - 2\operatorname{erf}\left(\frac{x}{\sqrt{4t}}\right) + \operatorname{erf}\left(\frac{x-1}{\sqrt{4t}}\right)\right]$$